Intuitionistic Logic Model Theory and Forcing

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INTRODUCTION

In 1963 P. Cohen established various fundamental independence results in set theory using a new technique which he called *forcing.* Since then there has been a deluge of new results of various kinds in set theory, proved using forcing techniques. It is a powerful method. It is, however, a method which is not as easy to interpret intuitively as the corresponding method of Gödel which establishes consistency results. Gödel defines an intuitively meaningful transfinite sequence of (domains of) classical models M_α, defines the class L to be the union of the M_α over all ordinals α, and shows L is a classical model for set theory [4; see also 3]. He then shows the axiom of constructability, the generalized continuum hypothesis, and the axiom of choice are true over L, establishing consistency.

In this book we define transfinite sequences of S. Kripke's intuitionistic models [13] in a manner exactly analogous to that of Gödel in the classical case (in fact, the M_α sequence is a particular example). In a reasonable way we define a "*class*" model for each sequence, which is to be a limit model over all ordinals. We show all the axioms of set theory are intuitionistically valid in the class models. Finally we show there are particular such sequences which provide: a class model in which the negation of the axiom of choice is intuitionistically valid; a class model in which the axiom of choice and the negation of the continuum hypothesis are intuitionistically valid; a class model in which the axiom of choice, the generalized continuum hypothesis, and the negation of the axiom of [14] constructability are intuitionistically valid. From this the *classical* independence results are shown to follow.

The definition of the sequences of intuitionistic models will be seen to be essentially the same as the definition of forcing in [3]. The difference is in the point of view. In Cohen's book one begins with a set M which is a *countable* model for set theory and, using forcing, one constructs a second countable model N "on top of" M. Forcing enables one to "discuss" N in M even though N is not a sub-model of M. Various such N are constructed for the different independence results. Cohen points out [3, pp. 147, 148] that actually the proofs can be carried out without the need for a countable model, and without constructing any classical models; this is the point of view we take. It is the forcing relation itself that the center of attention [see 3, page 147], though now it has an intuitive interpretation.

A similar program has been carried out by Vopênka and others. [See the series of papers 22, 23, 24, 27, 6, 25, 7, 8, 26, 28]. The primary difference is that these use topological intuitionistic model theory while we use Kripke's, which is much closer in form to forcing. Also the Vopênka series uses Gödel-Bernays set theory and generalizes the F_α sequence, while we use Zermelo-Fraenkel set theory and generalize the M_α sequence. The Vopênka treatment involves substantial topological considerations which we replace by more "logical" ones.

This book is divided into two parts. In part I we present a thorough treatment of the Kripke intuitionistic model theory. Part II consists of the set theory work discussed above.

Most of the material in Part I is not original but it is collected together and unified for the first time. The treatment is self-contained. Kripke models are defined (in notation different from that of Kripke). Tableau proof systems are defined using *signed* formulas (due to R. Smullyan), a device which simplifies the treatment. Three completeness proofs are presented (one for an axiom system, two for tableau systems), one due to Kripke [13], one due independently to R. Thomason [21] and the author, and one due to the author. We present proofs of compactness and Löwenheim-Skolem theorems. Adapting a method of Cohen, we establish a few connections between classical and intuitionistic logic. In the propositional case we give the relationship between Kripke models and algebraic ones [16] (which provides a fourth completeness proof in the 15 propositional case). Finally we present Schütte's proof of the intuitionistic Craig interpolation lemma [17], adapted to Kleene's tableau system G3 as modified by the use of signed formulas. No attempt is made to use methods of proof acceptable to intuitionists.

Chapter 7 begins part II. In it we define the notion of an *intuitionistic* Zermelo-Fraenkel (ZF) model, and the intuitionistic generalization of the Gödel M_α sequence. Most of the chapter is devoted to showing the class models resulting from the sequences of intuitionistic models are intuitionistic ZF models. This result is demonstrated in rather complete detail, especially sections 8 through 13, not because the work is particularly difficult, but because such models are comparatively unfamiliar.

In chapter 8 the independence of the axiom of choice is shown.

In chapter 9 we show how ordinals and cardinals may be represented in the intuitionistic models, and establish when such representatives exist.

Chapter l0 establishes the independence of the continuum hypothesis.

In Chapter 11 we give a way to represent constructable sets in the intuitionistic models, and establish when such representatives exist.

Chapter 12 establishes the independence of the axiom of constructability.

Chapter 13 is a collection of various results. We establish a connection between the sequences of intuitionistic models and the classical M_α sequence. We give some conditions under which the axiom of choice and the generalized continuum hypothesis will be valid in the intuitionistic class models (thus completing chapters 10 and 12). Finally we present Vopênka's method for producing classical non-standard set theory models from the intuitionistic class models without countability requirements [26].

The set theory work to this point is self-contained, given a knowledge of the Gödel consistency proof ([4], in more detail [3]).

In chapter 14 we present Scott and Solovay's notion of boolean valued models for set theory [19]. We define an intuitionistic (or forcing) generalization of the R_α sequence (sets with rank) analogous to the Cohen generalization of the M_α sequence, and we establish some connections between intuitionistic and boolean valued models for set theory. 19

PART I – LOGIC

CHAPTER I

PROPOSITIONAL INTUITIONISTIC LOGIC

SEMANTICS

§ 1. Formulas

We begin with a denumberable set of propositional variables *A*, *B*, *C*, three binary connectives $\wedge$, $\vee$, $\rightarrow$, and one unary connective ~, together with left and right parentheses (,). We shall informally use square and curly brackets [,], {, } for parentheses, to make reading simpler. The notion of *well formed formula,* or simply formula, is given recursively by the following rules:

F0. If *A* is a propositional variable, *A* is a formula.
Fl. If *X* is a formula, so is ~*X*.
F2, 3, 4. If *X* and *Y* are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$.

Remark 1.1: A propositional variable will sometimes be called an *atomic formula.*

It can be shown that the formation of a formula is unique. That is, for any given formula X, one and only one of the following can hold:
(1). *X* is *A* for some propositional variable *A*.
(2). There is a unique formula *Y* such that *X* is ~*Y*.
(3). There is a unique pair of formulas *Y* and *Z* and a unique binary connective b ($\wedge$, $\vee$ or $\rightarrow$) such that X is (*Y* b *Z*). We make use of this uniqueness of decomposition but do not prove it here.
We shall omit writing outer parentheses in a formula when no confusion [20] can result. Until otherwise stated, we shall use *A*, *B* and *C* for propositional variables, and *X*, *Y* and *Z* to represent any formula.
The notion *of immediate subformula is* given by the following rules:
I0. *A* has no immediate subformula.
Il. ~*X* has exactly one immediate subformula: *X*.
I2, 3, 4. $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$ each has exactly two immediate subformulas: *X* and *Y*.
The notion of *subformula is* defined as follows:
S0. *X* is a subformula of *X*.
S1. If *X* is an immediate subformula of *Y*, then *X* is a subformula of *Y*.
S2. If *X* is a subformula of *Y* and *Y* is a subformula of *Z*, then *X* is a subformula of *Z*.
By the *degree* of a formula is meant the number of occurrences of logical connectives (~, $\wedge$, $\vee$, $\rightarrow$) in the formula.

§ 2. **Models and validity**

By a (*propositional intuitionistic*) *model* we mean an ordered triple $<\mathcal{G}, \mathcal{R}, \models>$, where $\mathcal{G}$ is a non-empty set, $\mathcal{R}$ is a transitive, reflexive relation on $\mathcal{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathcal{G}$ and formulas, satisfying the following conditions:
For any $\Gamma \in \mathcal{G}$

P0. If $\Gamma \models A$ and $\Gamma \mathcal{R} \Delta$ then $\Delta \models A$ (recall A is atomic).
P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.
P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.
P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta =\mid X$.
P4. $\Gamma \models (X \rightarrow Y)$ iff for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.

Remark 2.1: For $\Gamma \in \mathcal{G}$, by Γ^* we shall mean any $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$. Thus "for all Γ^*, $\varphi(\Gamma^*)$" shall mean "for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\varphi(\Delta)$"; and "there is a Γ^* such that $\varphi(\Gamma^*)$" shall mean "there is a $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$ and $\varphi(\Delta)$". Thus P3 and P4 can be written more simply as:
P3. $\Gamma \models \sim X$ iff for all Γ^*, $\Gamma^* =\mid X$.
P4. $\Gamma \models (X \rightarrow Y)$ iff for all Γ^*, if $\Gamma^* \models X$, then $\Gamma^* \models Y$.

A particular formula X is called *valid in the model* $<\mathcal{G}, \mathcal{R}, \models>$ if for all $\Gamma \in \mathcal{G}$, $\Gamma \models X$. X is called *valid* if X is valid in all models. We will show [21] later that the collection of all valid formulas coincides with the usual collection of propositional intuitionistic logic theorems.

When it is necessary to distinguish between validity in this sense and the more usual notion, we shall refer to the validity defined above as intuitionistic validity, and the usual notion as classical validity. This notion of an intuitionistic model is due to Saul Kripke, and is presented, in different notation, in [13]. See also [18]. Examples of models will be found in section 5, chapter 2.

§ 3. Motivation

Let $<\mathcal{G}, \mathcal{R}, \models>$ be a model. $\mathcal{G}$ is intended to be a collection of possible universes, or more properly, states of knowledge. Thus a particular Γ in $\mathcal{G}$ may be considered as a collection of (physical) facts known at a particular time. The relation $\mathcal{R}$ represents (possible) time succession. That is, given two states of knowledge Γ and Δ of $\mathcal{G}$, to say $\Gamma \mathcal{R} \Delta$ is to say: if we now know Γ, it is possible that later we will know Δ. Finally, to say $\Gamma \models X$ is to say: knowing Γ, we know X, or: from the collection of facts Γ, we may deduce the truth of X.

Under this interpretation condition P3 of the last section, for example, may be interpreted as follows: from the facts Γ we may conclude $\sim X$ if and only if from no possible additional facts can we conclude X.

We might remark that under this interpretation it would seem reasonable that if $\Gamma \models X$ and $\Gamma \mathcal{R} \Delta$, then $\Delta \models X$, that is, if from a certain amount of information we can deduce X, given additional information, we still can deduce X, or if at some time we know X is true, at any later time we still know X is true. We have required that this holds only for the case that X is atomic, but the other cases follow.

For other interpretations of this modeling, see the original paper [13]. For a different but closely related model theory in terms of forcing see [5].

§ 4. Some properties of models

***Lemma* 4.1:** Let $<\mathcal{G}, \mathcal{R}, \models>$ and $<\mathcal{G}, \mathcal{R}, \models'>$ be two models such that for any atomic formula A and any $\Gamma \in G$, $\Gamma \models A$ iff $\Gamma \models' A$. Then $\models$ and $\models'$ are identical. [22]

Proof: We must show that for any formula X,

$$\Gamma \models X \Leftrightarrow \Gamma \models' X.$$

This is done by induction on the degree of X and is straightforward. We present one case as an example.

Suppose X is $\sim Y$ and the result is known for all formulas of degree less than that of X (in particular for Y). We show it for X:

$$\begin{aligned} \Gamma \models X &\Leftrightarrow \Gamma \models \sim Y \text{ (by definition)} \\ &\Leftrightarrow (\forall \Gamma^*)\ (\Gamma^* =\mid Y) \text{ (by hypothesis)} \\ &\Leftrightarrow (\forall \Gamma^*)\ (\Gamma^* =\mid' Y) \text{ (by definition)} \\ &\Leftrightarrow \Gamma \models' \sim Y \\ &\Leftrightarrow \Gamma \models' X. \end{aligned}$$

***Lemma* 4.2:** Let $\mathcal{G}$ be a non-empty set and $\mathcal{R}$ be a transitive, reflexive relation on $\mathcal{G}$. Suppose $\models$ is a relation between elements of $\mathcal{G}$ and *atomic* formulas. Then $\models$ can be extended to a relation $\models'$ between elements of G and *all* formulas in such a way that $<\mathcal{G}, \mathcal{R}, \models'>$ is a model.

Proof: We define $\models'$ as follows:

(0). if $\Gamma \models A$ then $\Gamma^* \models' A$,

(1). $\Gamma \models' (X \wedge Y)$ if $\Gamma \models' X$ and $\Gamma \models' Y$,

(2). $\Gamma \models' (X \vee Y)$ if $\Gamma \models' X$ or $\Gamma \models' Y$,

(3). $\Gamma \models' \sim X$ if for all Γ^*, $\Gamma^* =\mid' X$,

(4). $\Gamma \models' (X \rightarrow Y)$ if for all Γ^*, if $\Gamma^* \models' X$, then $\Gamma^* \models' Y$.

This is an inductive definition, the induction being on the degree of the formula. It is straightforward to show that $<\mathcal{G}, \mathcal{R}, \models'>$ is a model.

From lemmas 4.1 and 4.2 we immediately have

***Theorem* 4.3:** Let $\mathcal{G}$ be a non-empty set and be $\mathcal{R}$ a transitive, reflexive relation on $\mathcal{G}$. Suppose $\models$ is a relation between elements of $\mathcal{G}$ and atomic formulas. Then $\models$ can be extended in one and only one way to a relation, also denoted by $\models$, between elements of $\mathcal{G}$ and formulas, such that $<\mathcal{G}, \mathcal{R}, \models>$ is a model.

***Theorem* 4.4:** Let $<\mathcal{G}, \mathcal{R}, \models>$ be a model, X a formula and $\Gamma, \Delta \in \mathcal{G}$. If $\Gamma \models X$ and $\Gamma \mathcal{R} \Delta$, then $\Delta \models X$.

Proof: A straightforward induction on the degree of X (it is known already for X atomic). For example, suppose the result is known for X, and $\Gamma \models \sim X$. By definition, for all Γ^*, $\Gamma^* =\mid X$. But $\Gamma \mathcal{R} \Delta$ and $\mathcal{R}$ is transitive so any $\mathcal{R}$-successor of Δ is an $\mathcal{R}$-successor of Γ. Hence for all Δ^*, $\Delta^* =\mid X$, so $\Delta \models \sim X$. The other cases are similar. [23]

§ 5. Algebraic models

In addition to the Kripke intuitionistic semantics presented above, there is an older algebraic semantics: that of pseudo-boolean algebras. In this section we state the algebraic semantics, and in the next we prove its equivalence with Kripke's semantics. A thorough treatment of pseudoboolean algebras may be found in [16].

Definition 5.1: A *pseudo-boolean algebra* (PBA) is a pair $<\mathcal{B}, \leq>$ where $\mathcal{B}$ is a non-empty set and $\leq$ is a partial ordering relation on $\mathcal{B}$ such that for any two elements a and b of $\mathcal{B}$:

(1). the least upper bound $(a \cup b)$ exists.

(2). the greatest lower bound ($a \cap b$) exists.

(3). the pseudo complement of a relative to b ($a \Rightarrow b$), defined to be the largest $x \in \mathcal{B}$ such that $a \cap x \leq b$, exists.

(4). a least element $\wedge$ exists.

Remark 5.2: In the context $\Rightarrow$ is a mathematical symbol, not a metamathematical one.

Let $-a$ be $a \Rightarrow \wedge$ and $\vee$ be $-\wedge$.

Definition 5.3: h is called a *homomorphism* (from the set $\boldsymbol{W}$ of formulas to the PBA $<\mathcal{B}, \leq>$) if h: $\boldsymbol{W} \rightarrow \mathcal{B}$ and

(1). $h(X \wedge Y) = h(X) \cap h(Y)$,

(2). $h(X \vee Y) = h(X) \cup h(Y)$,

(3). $h(\sim X) = -h(X)$,

(4). $h(X \rightarrow Y) = h(X) \Rightarrow h(Y)$.

If $<\mathcal{B}, \leq>$ is a PBA and h is a homomorphism, the triple $<\mathcal{B}, \leq, h>$ is called an (*algebraic*) *model* for the set of formulas $\boldsymbol{W}$. If X is a formula, X is called (*algebraically*) *valid in the model* $<\mathcal{B}, \leq, h>$ if $h(X) = \vee$. X is *called* (*algebraically*) *valid* if X is valid in every model.

A proof may be found in [16] that the collection of all algebraically valid formulas coincides with the usual collection of intuitionistic theorems.

§ 6. Equivalence of algebraic and Kripke validity

First let us suppose we have a Kripke model $<\mathcal{G}, \mathcal{R}, \models>$ (we will not use the name "Kripke model" beyond this section). We will define an algebraic [24] model $<\mathcal{B}, \leq, h>$ such that for any formula X

$$h(X) = \vee \text{ iff for all } \Gamma \in \mathcal{G}, \Gamma \models X.$$

Remark 6.1: The following proof is based on exercise LXXXVI of [2].

Definition 6.1: If $b \subseteq \mathcal{G}$, we call b *$\mathcal{R}$-closed* if whenever $\Gamma \in b$ and $\Gamma \mathcal{R} \Delta$, then $\Delta \in b$.

We take for $\mathcal{B}$ the collection of all R-closed subsets of $\mathcal{G}$. For the ordering relation $\leq$ we take set inclusion $\subseteq$. Finally we define h by

$$h(X) = \{\Gamma \in \mathcal{G} \mid \Gamma \models X\}.$$

It is fairly straightforward to show that $<\mathcal{B}, \leq>$ is a PBA. Of the four required properties, the first two are left to the reader. We now show:

If $a, b \in \mathcal{B}$, there is a largest $x \in \mathcal{B}$ such that $a \cap x \leq b$.

We first note that the operations $\cup$ and $\cap$ are just the ordinary union and intersection. Now let p be the largest $\mathcal{R}$-closed subset of $(\mathcal{G} \div a) \cup b$ (where by $\div$ we mean ordinary set complementation). We will show that for all $x \in \mathcal{B}$

$$x \leq p \text{ iff } a \cap x \leq b,$$

which suffices.

Suppose $x \leq p$. Then

$$x \subseteq (\mathcal{G} \div a) \cup b,$$
$$a \cap x \subseteq a \cap [(\mathcal{G} \div a) \cup b],$$
$$a \cap x \subseteq a \cap b,$$
$$a \cap x \subseteq b,$$
$$a \cap x \leq b.$$

Conversely suppose $a \cap x \leq b$. Then

$$(a \cap x) \cup (x \div a) \subseteq b \cup (a \div x)$$
$$x \subseteq b \cup (a \div x)$$
$$x \subseteq b \cup (\mathcal{G} \div a)$$

but $x \in \mathcal{B}$, so x is $\mathcal{R}$-closed. Hence

$$x \subseteq p,$$
$$x \leq p.$$

The reader may verify that $\varnothing \in \mathcal{B}$ and is a least element.

Next we remark that h is a homomorphism. We demonstrate only one of the four cases, case (4). Thus we must show that $h(X \rightarrow Y)$ is the largest [25] $x \in \mathcal{B}$ such that

$$h(X) \cap x \leq h(Y).$$

First we show

$$h(X) \cap h(X \rightarrow Y) \leq h(Y),$$

that is

$$\{\Gamma \mid \Gamma \models X\} \cap \{\Gamma \mid \Gamma \models X \rightarrow Y\} \subseteq \{\Gamma \mid \Gamma \models Y\}.$$

But it is clear from the definition that

$$\text{if } \Gamma \models X \text{ and } \Gamma \models X \rightarrow Y, \text{ then } \Gamma \models Y.$$

Next suppose there is some $b \in \mathcal{B}$ such that $h(X) \cap b \leq h(Y)$ but $h(X \rightarrow Y) < b$. Then there must be some $\Gamma \in \mathcal{G}$ such that $\Gamma \in b$ but $\Gamma \notin h(X \rightarrow Y)$, i.e. $\Gamma =\mid X \rightarrow Y$. Since $\Gamma =\mid X \rightarrow Y$, there must be some Γ^* such that $\Gamma^* \models X$ but $\Gamma^* =\mid Y$. Since b is $\mathcal{R}$-closed, $\Gamma^* \in b$. But also $\Gamma^* \in h(X)$, so $\Gamma^* \in h(X) \cap b$, and so by assumption $\Gamma^* \in h(Y)$, that is $\Gamma^* \models Y$, a contradiction. Thus $h(X \rightarrow Y)$ is largest.

Thus $<\mathcal{B}, \leq, h>$ is an algebraic model. We leave it to the reader to verify that the unit element $\vee$ of $\mathcal{B}$ is $\mathcal{G}$ itself. Hence

$$h(X) = \vee \text{ iff for all } \Gamma \in \mathcal{G}, \Gamma \models X.$$

Conversely, suppose we have an algebraic model $<\mathcal{B}, \leq, h>$. We will define a Kripke model $<\mathcal{G}, \mathcal{R}, \models>$ so that for any formula X

$$h(X) = \vee \text{ iff for all } \Gamma \in \mathcal{G}, \Gamma \models X.$$

***Lemma* 6.2:** Let $\mathcal{F}$ be a filter in B and suppose $(a \Rightarrow b) \notin \mathcal{F}$. Then the filter generated by $\mathcal{F}$ and a does not contain b.

Proof: If the filter generated by $\mathcal{F}$ and a contained b, then ([16] p. 46, 8.2) for some $c \in \mathcal{F}$, $c \cap a \leq b$. So $c \leq (a \Rightarrow b)$ and hence $(a \Rightarrow b) \in \mathcal{F}$ by [16], p. 46, 8.2 again.

***Lemma* 6.3:** Let $\mathcal{F}$ be a proper filter in $\mathcal{B}$ and suppose $-a \notin \mathcal{F}$. Then the filter generated by $\mathcal{F}$ and a is also proper.

Proof: By lemma 6.2, since $-a = (a \Rightarrow \wedge)$.

***Lemma* 6.4:** Let $\mathcal{F}$ be a filter in $\mathcal{B}$ and suppose $a \notin \mathcal{F}$. Then $\mathcal{F}$ can be extended to a prime filter $\mathcal{P}$ such that $a \notin \mathcal{P}$.

Proof: (This is a slight modification of [16], p. 49, 9.2, included for completeness.) Let $\boldsymbol{O}$ be the collection of all filters in $\mathcal{B}$ not containing a. $\boldsymbol{O}$ is partially ordered by $\subseteq$. $\boldsymbol{O}$ is non-empty since $\mathcal{F} \in \boldsymbol{O}$. [26] Any chain in $\boldsymbol{O}$ has an upper bound since the union of any chain of filters is a filter. So by Zorn's lemma $\boldsymbol{O}$ contains a maximal element P. Of course $a \notin \mathcal{P}$. We need only show $\mathcal{P}$ is prime.

Suppose $\mathcal{P}$ is not prime. Then for some $a_1, a_2 \in \mathcal{B}$

$$a_1 \cup a_2 \in \mathcal{P}, a_1 \notin \mathcal{P}, a_2 \notin \mathcal{P}.$$

Let $\mathcal{S}_1$ be the filter generated by $\mathcal{P}$ and a_1, and $\mathcal{S}_2$ be the filter generated by $\mathcal{P}$ and a_2.

Suppose $a \in \mathcal{S}_1$ and $a \in \mathcal{S}_2$. Then [16, p. 46, 8.2] for some $c_1, c_2 \in \mathcal{P}$, $a_1 \cap c_1 \leq a$ and $a_2 \cap c_2 \leq a$. So for $c = c_1 \cap c_2$, $a_1 \cap c \leq a$ and $a_2 \cap c \leq a$, hence $(a_1 \cup a_2) \cap c \leq a$. But $c \in \mathcal{P}$ and $(a_1 \cup a_2) \in \mathcal{P}$, so $a \in \mathcal{P}$. But $a \notin \mathcal{P}$, so either $a \notin \mathcal{S}_1$ or $a \notin \mathcal{S}_2$.

Suppose $a \notin \mathcal{S}_1$. By definition $\mathcal{S}_1 \in \boldsymbol{O}$. But $\mathcal{S}_1$ is the filter generated by $\mathcal{P}$ and a_1, hence $\mathcal{P} \subseteq \mathcal{S}_1$. So $\mathcal{P}$ is not maximal, a contradiction. Similarly if $a \notin \mathcal{S}_2$. Thus $\mathcal{P}$ is prime.

Now we proceed with the main result. Recall that we have $<\mathcal{B}, \leq, h>$. Let $\mathcal{G}$ be the collection of all proper prime filters in $\mathcal{B}$. Let $\mathcal{R}$ be set inclusion $\subseteq$. For any $\Gamma \in G$ and any formula X, let $\Gamma \models X$ if $h(X) \in \Gamma$.

To show the resulting structure $<\mathcal{G}, \mathcal{R}, \models>$ is a model, we note property P0 is immediate. To show P1:

$$\begin{aligned} \Gamma \models (X \wedge Y) &\text{ iff } h(X \wedge Y) \in \Gamma \\ &\text{ iff } h(X) \cap h(Y) \in \Gamma \\ &\text{ iff } h(X) \in \Gamma \text{ and } h(Y) \in \Gamma \\ &\text{ iff } \Gamma \models X \text{ and } \Gamma \models Y \end{aligned}$$

(using the facts that h is a homomorphism and Γ is a filter). Similarly we show P2 using the fact that Γ is prime. To show P3:

Suppose $\Gamma \models \sim X$. Then $h(\sim X) \in \Gamma$ so

$$(\forall \Delta \in \mathcal{G}) (\Gamma \subseteq \Delta \text{ implies } h(\sim X) \in \Delta),$$
$$(\forall \Delta \in \mathcal{G}) (\Gamma \subseteq \Delta \text{ implies } h(X) \notin \Delta),$$

$$(\forall \Delta \in \mathcal{G})\ (\Gamma \mathcal{R} \Delta \text{ implies } \Delta =\!| X),$$

i.e. for all Γ^*, $\Gamma^* =\!| X$ (using the fact that $h(\sim X) \in \Delta$ and $h(X) \in \Delta$ imply $-h(X) \cap h(X) \in \Delta$, so $\wedge \in \Delta$ and Δ is not proper).

Suppose $\Gamma =\!| \sim X$. Then $h(\sim X) \notin \Gamma$, or $-h(X) \notin \Gamma$. By lemma 6.3 the filter generated by Γ and $h(X)$ is proper. By lemma 6.4 this filter can be [27] extended to a proper prime filter Δ. Then $\Gamma \subseteq \Delta$ and $h(X) \in \Delta$. So $(\exists \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta$ and $\Delta |\!= X)$, i.e. for some Γ^*, $\Gamma^* |\!= X$.

P4 is shown in the same way, but using lemma 6.2 instead of lemma 6.3. Thus $<\mathcal{G}, \mathcal{R}, |\!=>$ is a model.

Finally, to establish the desired equivalence, suppose first $h(X) = \vee$. Since $\vee$ is an element of every filter, for all $\Gamma \in \mathcal{G}$, $\Gamma |\!= X$. Conversely suppose $h(X) \neq \vee$. But $\{\vee\}$ is a filter and $h(X) \notin \{\vee\}$. By lemma 6.4 we can extend $\{\vee\}$ to a proper prime filter Γ such that $h(X) \notin \Gamma$. Thus $\Gamma \in \mathcal{G}$ and $\Gamma =\!| X$.

Thus we have shown

***Theorem* 6.5**: X is Kripke valid if and only if X is algebraically valid. [28]

CHAPTER 2

PROPOSITIONAL INTUITIONISTIC LOGIC

PROOF THEORY

§ 1. Beth tableaus

In this section we present a modified version of a proof system due originally to Beth. It is based on [2, § 145], but at the suggestion of R. Smullyan, we have introduced signed formulas and single trees in place of the unsigned formulas and dual trees of Beth.

By a *signed formula* we mean TX or FX where X is a formula. If $\boldsymbol{S}$ is a set of signed formulas and H is a single signed formula, we will write $\boldsymbol{S} \cup \{H\}$ simply as $\{\boldsymbol{S}, H\}$ or sometimes $\boldsymbol{S}, H$.

First we state the *reduction rules,* then we describe their use; $\boldsymbol{S}$ is any set (possibly empty) of signed formulas, and X and Y are any formulas:

$$T\wedge:\ \frac{\boldsymbol{S}, T(X \wedge Y)}{\boldsymbol{S}, TX, TY} \qquad F\wedge:\ \frac{\boldsymbol{S}, F(X \wedge Y)}{\boldsymbol{S}, FX \mid \boldsymbol{S}, FY}$$

$$T\vee:\ \frac{\boldsymbol{S}, T(X \vee Y)}{\boldsymbol{S}, TX \mid \boldsymbol{S}, TY} \qquad F\vee:\ \frac{\boldsymbol{S}, F(X \vee Y)}{\boldsymbol{S}, FX, FY}$$

$$T\sim:\ \frac{\boldsymbol{S}, T(\sim X)}{\boldsymbol{S}, FX} \qquad F\sim:\ \frac{\boldsymbol{S}, F(\sim X)}{\boldsymbol{S}_T, TX}$$

$$T\rightarrow:\ \frac{\boldsymbol{S}, T(X \rightarrow Y)}{\boldsymbol{S}, FX \mid \boldsymbol{S}, TY} \qquad F\rightarrow:\ \frac{\boldsymbol{S}, F(X \rightarrow Y)}{\boldsymbol{S}_T, TX, FY}$$

In rules $F\sim$ and $F\rightarrow$ above, $\boldsymbol{S}_T$ means $\{TX \mid TX \in \boldsymbol{S}\}$. [29]

Remark 1.1: $\boldsymbol{S}$ is a set, and hence $\{\boldsymbol{S}, TX\}$ is the same as $\{\boldsymbol{S}, TX, TX\}$. Thus duplication and elimination rules are not necessary.

If $\boldsymbol{U}$ is a set of signed formulas, we say one of the above rules, call it rule R, applies to $\boldsymbol{U}$ if by appropriate choice of $\boldsymbol{S}$, X and Y the collection of signed formulas above the line in rule R becomes $\boldsymbol{U}$.

By an *application of rule* R *to the set* $\boldsymbol{U}$ we mean the replacement of $\boldsymbol{U}$ by $\boldsymbol{U}_1$ (or by $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ if R is $F\wedge$, $T\vee$ or $T\rightarrow$) where $\boldsymbol{U}$ is the set of formulas above the line in rule R (after suitable substitution for $\boldsymbol{S}$, X and Y) and $\boldsymbol{U}_1$ (or $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ is the set of formulas below. This assumes R applies to $\boldsymbol{U}$. Otherwise the result is again $\boldsymbol{U}$. For example, by applying rule $F\rightarrow$ to the set $\{TX, FY, F(Z \rightarrow W)\}$ we may get the set $\{TX, TZ, FW\}$. By applying rule $T\vee$ to the set $\{TX, FY, T(Z \vee W)\}$ we may get the two sets $\{TX, FY, TZ\}$ and $\{TX, FY, TW\}$.

By a *configuration* $\mathcal{C}$ we mean a finite collection $\{\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n\}$ of sets of signed formulas.

By an *application of the rule* R *to the configuration* $\{\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n\}$ we mean the replacement of this configuration with a new one which is like the first except for containing instead of some $\boldsymbol{S}_i$ the result (or results) of applying rule R to $\boldsymbol{S}_i$.

By a *tableau* we mean a finite sequence of configurations $C_1, C_2, \ldots, C_n$ in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.

A set $\boldsymbol{S}$ of signed formulas is *closed* if it contains both TX and FX for some formula X. A configuration $\{\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n\}$ is closed if each $\boldsymbol{S}_i$, in it is closed. A tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ is closed if some $\mathcal{C}_i$ in it is closed.

By a *tableau for a set* $\boldsymbol{S}$ *of signed formulas we* mean a tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, in which $\mathcal{C}_1$ is $\{\boldsymbol{S}\}$. A finite set of signed formulas $\boldsymbol{S}$ is *inconsistent* if some tableau for $\boldsymbol{S}$ is closed. Otherwise $\boldsymbol{S}$ is *consistent.*

X is a *theorem* if $\{FX\}$ is inconsistent, and a closed tableau for $\{FX\}$ is called a *proof of* X. If X is a theorem we write $\vdash_I X$.

We will show in the next few sections the correctness and completeness of the above system relative to the semantics of ch. 1.

Examples of proofs in this system may be found in § 5.

The corresponding classical tableau system is like the above, but in rules $F\sim$ and $F\rightarrow$, $\boldsymbol{S}_T$ is replaced by $\boldsymbol{S}$ (see [20]). The interpretations of the classical and intuitionistic systems are different. [30]

In the classical system TX and FX mean X is true and X is false respectively. The rules may be read: if the situation above the line is the case, the situation below the line is also (or one of them is, if the rule is disjunctive: $F\wedge$, $T\vee$, $T\rightarrow$). Thus TX means the same as X, and FX means $\sim X$. Classically the signs T and F are dispensable. Proof is a refutation procedure. Suppose X is not true (begin a tableau with FX). Conclude that some formula must be both true and not true (a closed configuration is reached). Since this can not happen, X is true.

In the intuitionistic case TX is to mean X is known to be true (X is proven). FX is to mean X is not known to be true (X has not been proved). The rules are to be read: if the situation above the line is the case, then the situation below the line is possible, i.e. compatible with our present knowledge (if the rule is disjunctive, one of the situations below the line must be possible). For example consider rule $F\rightarrow$. If we have not proved $X \rightarrow Y$, it is possible to prove X without proving Y, for if this were not possible, a proof of Y would be "inherent" in a proof of X, and this fact would constitute a proof of $X \rightarrow Y$. But we have $\boldsymbol{S}_T$ below the line in this rule and not $\boldsymbol{S}$ because in proving X we might inadvertently verify some additional previously unproven formula (some $FZ \in \boldsymbol{S}$ might become TZ). Similarly for $F\sim$. The proof procedure is again by refutation. Suppose X is not proven (begin a tableau with FX). Conclude that it is possible that some formula is both proven and not proven. Since this is impossible, X is proven.

We have presented this system in a very formal fashion because it makes talking about it easier. In practice there are many simplifications which will become obvious in any attempt to use the method. Also, proofs may be written in a tree form. We find the resulting simplified system the easiest to use of all the intuitionistic proof systems, except in some cases, the system resulting by the same simplifications from the closely, related one presented in ch. 6 § 4. A full treatment of the corresponding classical tableau system, with practical simplifications, may be found in [20].

§ 2. Correctness of Beth tableaus

Definition 2.1: We call a set of signed formulas

$$\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$$ [31]

realizable if there is some model $<\mathcal{G}, \mathcal{R}, |= >$ and some $\Gamma \in \mathcal{G}$ such that $\Gamma |= X_1, \ldots, \Gamma |= X_n, \Gamma =| Y_1, \ldots, \Gamma =| Y_m$. We say that Γ *realizes* the set.

If $\{\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n\}$ is a configuration, we call it *realizable* if some $\boldsymbol{S}_i$ in it is realizable.

***Theorem* 2.2:** Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be a tableau. If $\mathcal{C}_i$ is realizable, so is $\mathcal{C}_{i+1}$.

Proof: We have eight cases, depending on the rule whose application produced $\mathcal{C}_{i+1}$ from $\mathcal{C}_i$.

Case (1): $\mathcal{C}_i$ is $\{\ldots, \{\boldsymbol{S}, T(X \vee Y)\}, \ldots\}$ and $\mathcal{C}_{i+1}$ is $\{\ldots, \{\boldsymbol{S}, TX\}, \{\boldsymbol{S}, TY\}, \ldots\}$. Since $\mathcal{C}_i$ is realizable, some element of it is realizable. If that element is not $\{\boldsymbol{S}, T(X \vee Y)\}$, the same element of $\mathcal{C}_{i+1}$. is realizable. If that element is $\{\boldsymbol{S}, T(X \vee Y)\}$, then for some model $<\mathcal{G}, \mathcal{R}, |= >$ and some $\Gamma \in \mathcal{G}$, Γ realizes $\{\boldsymbol{S}, T(X \vee Y)\}$. That is, Γ realizes $\boldsymbol{S}$ and $\Gamma |= (X \vee Y)$. Then $\Gamma |= X$ or $\Gamma |= Y$, so either Γ realizes $\{\boldsymbol{S}, TX\}$ or $\{\boldsymbol{S}, TY\}$. In either case $\mathcal{C}_{i+1}$ is realizable.

Case (2): $\mathcal{C}_i$ is $\{\ldots, \{\boldsymbol{S}, F(\sim X)\}, \ldots\}$ and $\mathcal{C}_{i+1}$ is $\{\ldots, \{\boldsymbol{S}_T, TX\}, \ldots\}$. $\mathcal{C}_i$ is realizable, and it suffices to consider the case that $\{\boldsymbol{S}, F(\sim X)\}$ is the realizable element. Then there is a model $<\mathcal{G}, \mathcal{R}, |= >$ and a $\Gamma \in \mathcal{G}$ such that Γ realizes $\boldsymbol{S}$ and $\Gamma =| \sim$X. Since $\Gamma =| \sim X$, for some $\Gamma^* \in \mathcal{G}$, $\Gamma^* |= X$. But clearly, if Γ realizes $\boldsymbol{S}$, Γ^* realizes $\boldsymbol{S}_T$ (by theorem 1.4.4). Hence Γ^* realizes $\{\boldsymbol{S}_T, TX\}$ and $\mathcal{C}_{i+1}$ is realizable.

The other six cases are similar.

***Corollary* 2.3**: The system of Beth tableaus is correct, that is, if $\vdash_I X$, X is valid.

Proof: We show the contrapositive. Suppose X is not valid. Then there is a model $<\mathcal{G}, \mathcal{R}, |= >$ and a $\Gamma \in \mathcal{G}$ such that $\Gamma =| X$. In other words $\{FX\}$ is realizable. But a proof of X would be a closed tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which $\mathcal{C}_1$ is $\{\{FX\}\}$. But $\mathcal{C}_1$ is realizable, hence each $\mathcal{C}_i$ is realizable. But obviously a realizable configuration cannot be closed. Hence $\dashv_I X$.

§ 3. Hintikka collections

In classical logic a set $\boldsymbol{S}$ of signed formulas is sometimes called downward saturated, or a Hintikka set, if

$TX \wedge Y \in \boldsymbol{S} \Rightarrow TX \in \boldsymbol{S}$ and $TY \in \boldsymbol{S}$,
$FX \vee Y \in \boldsymbol{S} \Rightarrow FX \in \boldsymbol{S}$ and $FY \in \boldsymbol{S}$, [32]
$TX \vee Y \in \boldsymbol{S} \Rightarrow TX \in \boldsymbol{S}$ or $TY \in \boldsymbol{S}$,
$FX \wedge Y \in \boldsymbol{S} \Rightarrow FX \in \boldsymbol{S}$ or $FY \in \boldsymbol{S}$,
$T{\sim}X \in \boldsymbol{S} \Rightarrow FX \in \boldsymbol{S}$,
$TX \rightarrow Y \in \boldsymbol{S} \Rightarrow FX \in \boldsymbol{S}$ or $TY \in \boldsymbol{S}$,
$F{\sim}X \in \boldsymbol{S} \Rightarrow TX \in \boldsymbol{S}$,
$FX \rightarrow Y \in \boldsymbol{S} \Rightarrow TX \in \boldsymbol{S}$ and $FY \in \boldsymbol{S}$.

Remark 3.1: The names Hintikka set and downward saturated set were given by Smullyan [20]. Hintikka, their originator, called them model sets.

Hintikka showed that any consistent downward saturated set could be included in a set for which the above properties hold with ⇒replaced by ⇔. From this follows the completeness of certain classical tableau systems. This approach is thoroughly developed by Smullyan in [20].

We now introduce a corresponding notion in intuitionistic logic, which we call a Hintikka collection. While its intuitive appeal may not be as immediate as in the classical case, its usefulness is as great.

Definition 3.2: Let $\mathcal{G}$ be a collection of consistent sets of signed formulas. We call $\mathcal{G}$ a *Hintikka collection* if for any $\Gamma \in \mathcal{G}$

$$
\begin{aligned}
TX \wedge Y \in \Gamma &\Rightarrow TX \in \Gamma \text{ and } TY \in \Gamma,\\
FX \vee Y \in \Gamma &\Rightarrow FX \in \Gamma \text{ and } FY \in \Gamma,\\
TX \vee Y \in \Gamma &\Rightarrow TX \in \Gamma \text{ or } TY \in \Gamma,\\
FX \wedge Y \in \Gamma &\Rightarrow FX \in \Gamma \text{ or } FY \in \Gamma,\\
T{\sim}X \in \Gamma &\Rightarrow FX \in \Gamma,\\
TX \rightarrow Y \in \Gamma &\Rightarrow FX \in \Gamma \text{ or } TY \in \Gamma,\\
F{\sim}X \in \Gamma &\Rightarrow \text{for some } \Delta \in G,\ \Gamma_T \subseteq \Delta \text{ and } TX \in \Delta,\\
FX \rightarrow Y \in \Gamma &\Rightarrow \text{for some } \Delta \in G,\ \Gamma_T \subseteq \Delta \text{ and } TX \in \Delta,\ FY \in \Delta .
\end{aligned}
$$

Definition 3.3**:** Let $\mathcal{G}$ be a Hintikka collection. We call $<\mathcal{G}, \mathcal{R}, \models>$ a *model for* $\mathcal{G}$ if
(1). $<\mathcal{G}, \mathcal{R}, \models>$ is a model,
(2). $\Gamma_T \subseteq \Delta \Rightarrow \Gamma\mathcal{R}\Delta$,
(3). $TX \in \Gamma \Rightarrow \Gamma \models X$, $FX \in \Gamma \Rightarrow \Gamma =\!| X$.

***Theorem* 3.4:** There is a model for any Hintikka collection.
Proof: Let $\mathcal{G}$ be a Hintikka collection. Define $\mathcal{R}$ by: $\Gamma\mathcal{R}\Delta$ if $\Gamma_T \subseteq \Delta$. [33] If A is atomic, let $\Gamma \models A$ if $TA \in \Gamma$, and extend $\models$ to produce a model $<\mathcal{G}, \mathcal{R}, \models>$. To show property (3) is a straightforward induction on the degree of X. We give one case as illustration. Suppose X is ${\sim}Y$ and the result is known for Y. Then

$$
\begin{aligned}
T{\sim}Y \in \Gamma &\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \Rightarrow T{\sim}X \in \Delta)\\
&\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \Rightarrow FY \in \Delta)\\
&\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma\mathcal{R}\Delta \Rightarrow \Delta =\!| Y)\\
&\Rightarrow \Gamma \models {\sim}Y,
\end{aligned}
$$

and

$$
\begin{aligned}
F{\sim}Y \in \Gamma &\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \text{ and } TY \in \Delta)\\
&\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma\mathcal{R}\Delta \text{ and } \Delta \models \mathrm{Y})\\
&\Rightarrow \Gamma =\!| {\sim}Y.
\end{aligned}
$$

It follows from this theorem that to show the completeness of Beth tableaus we need only show the following: If $\longrightarrow\!|_I X$, then there is a Hintikka collection $\mathcal{G}$ such that for some $\Gamma \in \mathcal{G}$, $FX \in \Gamma$.

§ 4. Completeness of Beth tableaus

Let ***S*** be a set of signed formulas. By $\mathcal{S}(\boldsymbol{S})$ we mean the collection of all signed subformulas of formulas in ***S***. If ***S*** is finite, $\mathcal{S}(\boldsymbol{S})$ is finite.

Let ***S*** be a finite, consistent set of signed formulas. We define a *reduced set for* ***S*** (there may be many) as follows:

Let $\boldsymbol{S}_0$ be $\boldsymbol{S}$. Having defined $\boldsymbol{S}$, a finite consistent set of signed formulas, suppose one of the following Beth reduction rules applies to $\boldsymbol{S}$: $T\wedge$, $F\wedge$, $T\vee$, $F\vee$, $T\sim$ or $T\rightarrow$. Choose one which applies, say $F\wedge$. Then $\boldsymbol{S}_n$ is $\{\boldsymbol{U}, FX \wedge Y\}$. This is consistent, so clearly either $\{\boldsymbol{U}, FX \wedge Y, FX\}$ or $\{\boldsymbol{U}, FX \wedge Y, FY\}$ is consistent. Let $\boldsymbol{S}_{n+1}$ be $\{\boldsymbol{U}, FX \wedge Y, FX\}$ if consistent, otherwise let $\boldsymbol{S}_{n+1}$ be $\{\boldsymbol{U}, FX \wedge Y, FY\}$. Similarly if $T\wedge$ applies and was chosen, then $\boldsymbol{S}_n$ is $\{\boldsymbol{U}, TX \wedge Y\}$. Since this is consistent, $\{\boldsymbol{U}, TX \wedge Y, TX, TY\}$ is consistent. Let this be $\boldsymbol{S}_{n+1}$. In this way we define a sequence $\boldsymbol{S}_0, \boldsymbol{S}_1, \boldsymbol{S}_2, \ldots$ This sequence has the property $\boldsymbol{S}_n \subseteq \boldsymbol{S}_{n+1}$. Further, each $\boldsymbol{S}_n$ is finite and consistent. Since each $\boldsymbol{S}_n \subseteq \mathcal{S}(\boldsymbol{S})$, there are only a finite number of different possible $\boldsymbol{S}_n$. Consequently there must be a member of the sequence, say $\boldsymbol{S}_n$, such that the application of any one of the rules (except $F\sim$ or $F\rightarrow$) produces $\boldsymbol{S}_n$ again. Call such an $\boldsymbol{S}_n$ a *reduced* set of $\boldsymbol{S}$, and denote it by $\boldsymbol{S'}$. Clearly any finite, consistent set of [34] signed formulas ha a finite, consistent reduced set. Moreover, if $\boldsymbol{S'}$ is a reduced set, it has the following suggestive properties:

$TX \wedge Y \in \boldsymbol{S'} \Rightarrow TX \in \boldsymbol{S'}$ and $TY \in \boldsymbol{S'}$,
$FX \vee Y \in \boldsymbol{S'} \Rightarrow FX \in \boldsymbol{S'}$ and $FY \in \boldsymbol{S'}$,
$TX \vee Y \in \boldsymbol{S'} \Rightarrow TX \in \boldsymbol{S'}$ or $TY \in \boldsymbol{S'}$,
$FX \wedge Y \in \boldsymbol{S'} \Rightarrow FX \in \boldsymbol{S'}$ or $FY \in \boldsymbol{S'}$,
$T{\sim}X \in \boldsymbol{S'} \Rightarrow FX \in \boldsymbol{S'}$,
$TX \rightarrow Y \in \boldsymbol{S'} \Rightarrow FX \in \boldsymbol{S'}$ or $TY \in \boldsymbol{S'}$,
$\boldsymbol{S'}$ is consistent.

Now, given any finite, consistent set of signed formulas S, we form the collection *of associated sets* as follows:

If $F{\sim}X \in \boldsymbol{S}$, $\{\boldsymbol{S}_T, TX\}$ is an associated set.
If $FX \rightarrow Y \in \boldsymbol{S}$, $\{\boldsymbol{S}_T, TX, FY\}$ is an associated set.

Let $\mathcal{A}(\boldsymbol{S})$ be the collection of all associated sets of $\boldsymbol{S}$. $\mathcal{A}(\boldsymbol{S})$ is finite, since $\boldsymbol{U} \in \mathcal{A}(\boldsymbol{S})$ implies $\boldsymbol{U} \subseteq \mathcal{S}(\boldsymbol{S})$ and $\mathcal{S}(\boldsymbol{S})$ is finite. $\mathcal{A}(\boldsymbol{S})$ has the following properties: if $\boldsymbol{S}$ is consistent, any associated set is consistent and

$F{\sim}X \in \boldsymbol{S} \Rightarrow$ for some $\boldsymbol{U} \in \mathcal{A}(\boldsymbol{S})$ $\boldsymbol{S}_T \subseteq \boldsymbol{U}$, $TX \in \boldsymbol{U}$,
$FX \rightarrow Y \in \boldsymbol{S} \Rightarrow$ for some $\boldsymbol{U} \in A(\boldsymbol{S})$ $\boldsymbol{S}_T \subseteq \boldsymbol{U}$, $TX \in \boldsymbol{U}$, $FY \in \boldsymbol{U}$.

Now we proceed with the proof of completeness.

Suppose $\dashv_I X$. Then $\{FX\}$ is consistent. Extend it to its reduced set $\boldsymbol{S}_0$. Form $\mathcal{A}(\boldsymbol{S}_0)$. Let the elements of d $A(\boldsymbol{S}_0)$ be $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_n$. Let $\boldsymbol{S}_1$ be the reduced set of $\boldsymbol{U}_1, \ldots, \boldsymbol{S}_n$ be the reduced set of $\boldsymbol{U}_n$. Thus, we have the sequence $\boldsymbol{S}_0, \boldsymbol{S}_1, S_2, \ldots, \boldsymbol{S}_n$.

Next form $\mathcal{A}(\boldsymbol{S}_1)$. Call its elements $\boldsymbol{U}_{n+1}, \boldsymbol{U}_{n+2}, \ldots \boldsymbol{U}_m$. Let $\boldsymbol{S}_{n+1}$ be the reduced set of $\boldsymbol{U}_{n+1}$ and so on. Thus, we have the sequence $\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots, \boldsymbol{S}_n, \boldsymbol{S}_{n+1}, \ldots, \boldsymbol{S}_m$. Now we repeat the process with $\boldsymbol{S}_2$, and so on.

In this way we form a sequence $\boldsymbol{S}_0, \boldsymbol{S}_1, \boldsymbol{S}_2, \ldots$. Since each $\boldsymbol{S}_i \subseteq \mathcal{S}(\boldsymbol{S})$, there are only finitely many possible different $\boldsymbol{S}_i$. Thus we must reach a point $\boldsymbol{S}_k$ of the sequence such that any continuation repeats on earlier member.

Let $\mathcal{G}$ be the collection $\{\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots, \boldsymbol{S}_k\}$. It is easy to see that $\mathcal{G}$ is a Hintikka collection. But $FX \in \boldsymbol{S}_0 \in \mathcal{G}$. Thus we have shown:

***Theorem* 4.1:** Beth tableaus are complete. [35]

Remark 4.2: This proof also establishes that propositional intuitionistic logic is decidable. For, if we follow the above procedure beginning with FX, after a finite number of steps we will have either a closed tableau for $\{FX\}$ or a counter-model for X. Moreover, the number of steps may be bounded in terms of the degree of X.

The completeness proof presented here is in essence the original proof of Kripke [13]. For a different tableau completeness proof see ch. 5 § 6, where it is given for first order logic. For a completeness proof of an axiom system see eh. 5 § 10, where it also is given for a first order system. The work in ch. I § 6 provides an algebraic completeness proof, since the Lindenbaum algebra of intuitionistic logic is easily shown to be a pseudo-boolean algebra. See [16].

§ **5. Examples**

In this section, so that the reader may gain familiarity with the foregoing, we present a few theorems and non-theorems of intuitionistic propositional logic, together with their proofs or counter-models.

We show

(1). $\dashv_{I} A \vee \sim A$,
(2). $\vdash_{I} \sim\sim(A \vee \sim A)$,
(3). $\dashv_{I} \sim\sim A \rightarrow A$,
(4). $\vdash_{I} (A \vee B) \rightarrow \sim(\sim A \wedge \sim B)$,
(5). $\dashv_{I} \sim\sim(A \vee B) \rightarrow (\sim\sim A \wedge \sim\sim B)$.

For the general principle connecting (1) and (2) see ch. 4 § 8.

(1). $\dashv_{I} A \vee \sim A$.

A counter example for this is the following:

$$\mathcal{G} = \{\Gamma, \Delta\}$$
$$\Gamma\mathcal{R}\Gamma,\ \Gamma\mathcal{R}\Delta,\ \Delta\mathcal{R}\Delta.$$

$\Delta \models A$ is the $\models$ relation for atomic formulas, and $\models$ is extended to all formulas as usual. We may schematically represent this model by

$$\begin{array}{c}\Gamma \\ | \\ \Delta \models A\end{array}$$ [36]

We claim $\Gamma =\mid A \vee \sim A$. Suppose not. If $\Gamma \models A \vee \sim A$, either $\Gamma \models A$ $\Gamma \models \sim A$. But $\Gamma =\mid A$. If $\Gamma \models \sim A$ then since $\Gamma\mathcal{R}\Delta$, $\Delta =\mid A$. But $\Delta \models A$, hence $\Gamma =\mid A \vee \sim A$.

(2). $\vdash_{I} \sim\sim(A \vee \sim A)$.

A tableau proof for this is the following, where the reasons for the steps are obvious:

$\{\{F\sim\sim(A \vee \sim A)\}\}$,
$\{\{T\sim(A \vee \sim A)\}\}$,

$\{\{T{\sim}(A \vee {\sim}A\}, F(A \vee {\sim}A\}\}$,
$\{\{T{\sim}(A \vee {\sim}A\}, FA, F{\sim}A\}\}$,
$\{\{T{\sim}(A \vee {\sim}A\}, TA\}\}$,
$\{\{F(A \vee {\sim}A\}, TA\}\}$,
$\{\{FA, F{\sim}A\}, TA\}\}$,

(3). $\dashv_{I} \sim{\sim}A \to A$.

The model of example (1) has the property that $\Gamma \models \sim{\sim}A$ but $\Gamma =\!\mid A$.

(4). $\vdash_{I} (A \vee B) \to \sim({\sim}A \wedge {\sim}B)$.

The following is a proof:

$\{\{F((A \vee B) \to \sim({\sim}A \wedge {\sim}B))\}\}$,
$\{\{T(A \vee B), F{\sim}({\sim}A \wedge {\sim}B)\}\}$,
$\{\{T(A \vee B), T({\sim}A \wedge {\sim}B)\}\}$,
$\{\{T(A \vee B), T{\sim}A, T{\sim}B\}\}$,
$\{\{T(A \vee B), FA, T{\sim}B\}\}$,
$\{\{T(A \vee B), FA, FB\}\}$,
$\{\{TA, FA, FB\}, \{TB, FA, FB\}\}$.

(5). $\vdash_{I} \sim{\sim}(A \vee B) \to (\sim{\sim}A \vee \sim{\sim}B)$.

A counter example is the following:

$\mathcal{G} = \{\Gamma, \Delta, \Omega\}$,
$\Gamma \mathcal{R} \Gamma$, $\Delta \mathcal{R} \Delta$, $\Omega \mathcal{R} \Omega$,
$\Gamma \mathcal{R} \Delta$, $\Gamma \mathcal{R} \Omega$.
$\Delta \models A$, $\Omega \models B$, is the $\models$ relation for atomic formulas, and is extended as usual. We may schematically represent this model by

$$\begin{array}{c} \Gamma \\ \swarrow \;\; \searrow \\ \Delta \models A \quad \Omega \models B \end{array}$$ [37]

Now $\Delta \models A$, so $\Delta \models A \vee B$. Likewise $\Omega \models A \vee B$. It follows that $\Gamma \models \sim{\sim}(A \vee B)$. But if $\Gamma \models \sim{\sim}A \vee \sim{\sim}B$, either $\Gamma \models \sim{\sim}A$ or $\Gamma \models \sim{\sim}B$. If $\Gamma \models \sim{\sim}A$, it would follow that $\Omega \models A$. If $\Gamma \models \sim{\sim}B$, it would follow that $\Delta \models B$. Thus $\Gamma =\!\mid \sim{\sim}A \vee \sim{\sim}B$. [38]

CHAPTER 3

RELATED SYSTEMS OF LOGIC

§ 1. *f*-primitive intuitionistic logic, semantics

This is an alternative formulation of intuitionistic logic in which a symbol f is taken as primitive, instead of , which is then re-introduced as a formal abbreviation, $\sim X$ for $X \rightarrow f$. For presentations of this type, see [15] or [17].

Specifically, we change the definition of formula by adding f to our list of propositional variables and removing ~ from the set of connectives. ~ is re-introduced as a metamathematical symbol as above. Our definition of subformula is also changed accordingly. The definition of model is changed as follows: replace P3 (ch. 1 § 2) by P3': $\Gamma =\!\mid f$. This leads to a new definition of validity, which we may call f-validity.

***Theorem* 1.1:** Let X be a formula (in the usual sense) and let X' be the corresponding formula with ~ written in terms of f. Then X is valid if and only if X' is f-valid.

Proof: We show that in any model $<\mathcal{G}, \mathcal{R}, \models>$

$$\Gamma \models X \text{ iff } \Gamma \models X'$$

(where we use two different senses of $\models$). The proof is by induction on the degree of X (which is the same as the degree of X'). Actually all cases [39] are easy except that of ~ itself. So suppose the result is known for all formulas of degree less than that of X, and X is $\sim Y$. Then

$$\begin{aligned} \Gamma \models X &\Leftrightarrow \Gamma \models \sim Y \\ &\Leftrightarrow \forall \Gamma^* \ \Gamma^* =\!\mid Y \\ &\Leftrightarrow \forall \Gamma^* \ \Gamma^* =\!\mid Y', \end{aligned}$$

but clearly this is equivalent to $\Gamma \models Y' \rightarrow f$ since $\Gamma^* =\!\mid f$. Hence equivalently $\Gamma \models X'$.

§ 2. *f*-primitive intuitionistic logic, proof theory

In this section we still retain the altered definition of formula in the last section with f primitive. We give a tableau system for this. The new system is the same as that of ch. 2 § 1 in all but two respects. First the rules $T\sim$ and $F\sim$ are removed. Second a set S of signed formulas is called closed if it contains TX and FX for some formula X, or if it contains Tf.

This leads to a new definition of theorem, which we may call f-theorem.

***Theorem* 2.1**: Let X be a formula (in the usual sense) and let X' be the corresponding formula with $\rightarrow$ written in terms of f. Then X is a theorem if and only if X' is an f-theorem.

This follows immediately from the following:

Lemma* 2.2:** Let ***S be a set of signed formulas (in the usual sense) and let ***S'*** be the corresponding set of signed formulas with ~ replaced in terms of f. Then ***S*** is inconsistent if and only if ***S'*** is *f*-inconsistent.

Proof: We show this in two halves. First suppose ***S*** is inconsistent. We show the result by induction on the length of the closed tableau for ***S***. There are only two significant cases. Suppose first that the tableau for ***S*** is $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$; $\mathcal{C}_1$ is {{ ***U***, *F*~*X*}} and $\mathcal{C}_2$ is{{ $\boldsymbol{U}_T$, *TX*}}. Then by the induction hypothesis {{ $\boldsymbol{U'}_T$, *TX'*}} is *f*-inconsistent. Hence so is {{ ***U'***, *FX'* → *f*}}, i.e. ***S'***. The other case is if $\mathcal{C}_1$ is {{***U***, *T*~*X*}} and $\mathcal{C}_2$ is {{***U***, *FX*)}. Then by the induction hypothesis {***U'***, *FX'*} is *f*-inconsistent. Hence so is {***U'***, *TX'*→ *f*}, i.e. ***S'***.

The converse is shown by induction on the length of the closed *f*-tableau for ***S'***. If this *f*-tableau is of length *l*, either ***S'*** contains *TX* and [40] *FX* for some formula *X*, and we are done, or ***S'*** contains *Tf*, which is not possible since we supposed ***S'*** arose from standard set ***S***.

The induction steps are similar to those above.

The results of this and the last sections, together with our earlier results give: *X'* is *f*-valid if and only if *X*' is an *f*-theorem. This is not the complete generality one would like since it holds only for those formulas *X'* which correspond to standard formulas *X*. The more complete result is however true, as the reader may show by methods similar to those of the last chapter.

§ 3. Minimal logic

Minimal logic is a sublogic of intuitionistic logic in which a false statement need not imply everything. The original paper on minimal logic is Johannson's [9]. Prawitz establishes several results concerning it in [15], and it is treated algebraically by Rasiowa and Sikorski [16].

Semantically, we use the *f*-models defined in § 1, with the change that we no longer require P3', that is, that $\Gamma =\!| f$. Proof theoretically, we use the *f*-tableaus defined in §2, with the change that we no longer have closure of a set because it contains *Tf*. We leave it to the reader to show that *X* is provable in this tableau system if and only if *X* is valid in this model sense, using the methods of ch. 2.

Certainly every minimal logic theorem is an intuitionistic logic theorem, but the converse is not true. For example $(A \wedge {\sim}A) \to B$ is a theorem of intuitionistic logic, but the following is a minimal counter-model for it, or rather for $(A \wedge (A \to f)) \to B$:

$$\mathcal{G} = \{\Gamma\}$$
$$\Gamma \mathcal{R} \Gamma$$
$$\Gamma \models A,\ \Gamma \models f,$$

and $\models$ is extended as usual. It is easily seen that $\Gamma \models A \wedge (A \to f)$, but $\Gamma =\!| B$.

§ 4. Classical logic

Beginning with this section, we return to the usual notions of formula, tableau and model, that is, with ~ and not *f* as primitive. [41]

Some authors call a set $\mathcal{S}$ of unsigned formulas a (classical) truth set if

$$X \wedge Y \in \mathcal{S} \Leftrightarrow X \in \mathcal{S} \text{ and } Y \in \mathcal{S},$$
$$X \vee Y \in \mathcal{S} \Leftrightarrow X \in \mathcal{S} \text{ or } Y \in \mathcal{S},$$
$${\sim}X \in \mathcal{S} \Leftrightarrow X \notin \mathcal{S},$$
$$X \rightarrow Y \in \mathcal{S} \Leftrightarrow X \notin \mathcal{S} \text{ or } Y \in \mathcal{S}.$$

It is a standard result of classical logic that X is a classical theorem if and only if X is in every truth set. There is a proof of this in [20].

***Theorem* 4.1:** Any intuitionistic theorem is a classical theorem.

Proof: Suppose X is not a classical theorem. Then there is a truth set $\mathcal{S}$ such that $X \notin \mathcal{S}$. We define a very simple intuitionistic counter-model for X, $<\mathcal{G}, \mathcal{R}, |\!=>$, as follows:

$$\mathcal{G} = \{\mathcal{S}\},$$
$$\mathcal{S}\mathcal{R}\mathcal{S},$$
$$\mathcal{S} |\!= A \Leftrightarrow A \in \mathcal{S},$$

for A atomic, and $|\!=$ is extended as usual. It is easily shown by induction on the degree of Y that

$$\mathcal{S} |\!= Y \Leftrightarrow Y \in \mathcal{S}.$$

Hence $\mathcal{S} =\!| X$, and X is not an intuitionistic theorem.

That the converse is not true follows since we showed in ch. 2 § 5 that $\dashv_{I} A \vee {\sim}A$. Thus we have: minimal logic is a proper sub-logic of intuitionistic logic which is a proper sub-logic of classical logic.

§ 5. Modal logic, S4; semantics

In this section we define the set of (propositional) S4 theorems semantically using a model due to Kripke [12] (see also [18]). S4 was originated by Lewis [14], and an algebraic treatment may be found in [16]. A natural deduction treatment is in [15].

The definition of formula is changed by adding $\Box$ to the set of unary connectives. Thus for example ${\sim}\Box{\sim}(A \vee \Box{\sim}A)$ is a formula. $\Box$ is read "necessarily". $\Diamond$ is sometimes taken as an abbreviation for ${\sim}\Box{\sim}$ and is read "possibly". (In [14] $\Diamond$ was primitive.)

The S4 model is defined as follows: It is an ordered triple $<\mathcal{G}, \mathcal{R}, |\!=>$ where $\mathcal{G}$ is a non-empty set, $\mathcal{R}$ is a transitive, reflexive relation on $\mathcal{G}$, [42] and $|\!=$ is a relation between elements of $\mathcal{G}$ and formulas, satisfying the following conditions:

M1. $\Gamma |\!= X \wedge Y$ iff $\Gamma |\!= X$ and $\Gamma |\!= Y$.
M2. $\Gamma |\!= X \vee Y$ iff $\Gamma |\!= X$ or $\Gamma |\!= Y$.
M3. $\Gamma |\!= {\sim}X$ iff $\Gamma =\!| X$.
M4. $\Gamma |\!= (X \rightarrow Y)$ iff $\Gamma =\!| X$ or $\Gamma |\!= Y$.
M5. $\Gamma |\!= \Box X$ iff for all Γ^*, $\Gamma^* |\!= X$.

X is S4 valid in $<\mathcal{G}, \mathcal{R}, |\!=>$ if for all $\Gamma \in \mathcal{G}$, $\Gamma |\!= X$. X is S4 valid if X is S4 valid in all S4 models.

The intuitive idea behind this modeling is the following: $\mathcal{G}$ is the collection of all possible worlds. $\Gamma\mathcal{R}\Delta$ means Δ is a world possible relative to Γ. $\Gamma \models X$ means X is true in the world Γ. Thus M5 may be interpreted: X is necessarily true in Γ if and only if X is true in any world possible relative to Γ. This interpretation is given in [12].

§ 6. Modal logic, S4; proof theory

We define a tableau system for S4 as follows: Everything in the definition of Beth tableaus in ch. 2 § i remains the same except the reduction rules themselves. These are replaced by

$$MT\wedge: \frac{\boldsymbol{S},\ TX \wedge Y}{\boldsymbol{S},\ TX,\ TY} \qquad MF\wedge: \frac{\boldsymbol{S},\ FX \wedge Y}{\boldsymbol{S},\ FX \mid \boldsymbol{S},\ FY}$$

$$MT\vee: \frac{\boldsymbol{S},\ TX \vee Y}{\boldsymbol{S},\ TX \mid \boldsymbol{S},\ TY} \qquad MF\vee: \frac{\boldsymbol{S},\ FX \vee Y}{\boldsymbol{S},\ FX,\ FY}$$

$$MT\sim: \frac{\boldsymbol{S},\ T{\sim}X}{\boldsymbol{S},\ FX} \qquad MF\sim: \frac{\boldsymbol{S},\ F{\sim}X}{\boldsymbol{S},\ TX}$$

$$MT{\rightarrow}: \frac{\boldsymbol{S},\ TX \rightarrow Y}{\boldsymbol{S},\ FX \mid \boldsymbol{S},\ TY} \qquad MF{\rightarrow}: \frac{\boldsymbol{S},\ FX{\rightarrow}Y}{\boldsymbol{S},\ TX,\ FY}$$

$$MT\Box: \frac{\boldsymbol{S},\ T\Box X}{\boldsymbol{S},\ TX} \qquad MF\Box: \frac{\boldsymbol{S},\ F\Box X}{\boldsymbol{S}_{\Box},\ FX}$$

where in rule $MF\Box$ $\boldsymbol{S}_{\Box}$ is $\{T\Box X \mid T\Box X \in \boldsymbol{S}\}$. Again the methods of ch. 2 can be adapted to S4 to establish the identity of the set of S4 theorems and the set of S4 valid formulas. This is left to the reader. The original [43] proof is in [12]. We are more interested in the relation between S4 and intuitionistic logic.

§ 7. S4 and intuitionistic logic

A map from the set of intuitionistic formulas to the set of S4 formulas is defined by (see [18])

$$\begin{aligned} \boldsymbol{M}(A) &= \Box A \text{ for } A \text{ atomic},\\ \boldsymbol{M}(X \vee Y) &= \boldsymbol{M}(X) \vee \boldsymbol{M}(Y),\\ \boldsymbol{M}(X \wedge Y) &= \boldsymbol{M}(X) \wedge \boldsymbol{M}(Y),\\ \boldsymbol{M}({\sim}X) &= \Box{\sim}\boldsymbol{M}(X),\\ \boldsymbol{M}(X \rightarrow Y) &= \Box(\boldsymbol{M}(X) \rightarrow M(Y)). \end{aligned}$$

We wish to show

***Theorem* 7.1:** If X is an intuitionistic formula, X is intuitionistically valid if and only if $\boldsymbol{M}(X)$ is S4-valid.

This follows from the next three lemmas.

***Lemma* 7.2:** Let $<\mathcal{G}, \mathcal{R}, |=_{\text{I}}>$ be an intuitionistic model and $<\mathcal{G}, \mathcal{R}, |=_{\text{S4}}>$ be an S4 model, such that for any $\Gamma \in \mathcal{G}$ and any atomic A

$$\Gamma |=_{\text{I}} A \Leftrightarrow \Gamma |=_{\text{S4}} A.$$

Then for any formula X

$$\Gamma |=_{\text{I}} X \Leftrightarrow \Gamma |=_{\text{S4}} X.$$

Proof: A straightforward induction on the degree of X.

***Lemma* 7.3:** Given an intuitionistic counter-model for X, there is an S4 counter-model for $\boldsymbol{M}(X)$.

Proof: We have $<\mathcal{G}, \mathcal{R}, |=_{\text{I}}>$, an intuitionistic model such that for some $\Gamma \in \mathcal{G}\, \Gamma |=_{\text{I}} X$. We take for our S4 model $<\mathcal{G}, \mathcal{R}, |=_{\text{S4}}>$, where $|=_{\text{S4}}$ is defined by

$$\Delta |=_{\text{S4}} A \text{ if } \Delta |=_{\text{I}} A$$

for A atomic and any Δ in $\mathcal{G}$, and $|=_{\text{S4}}$ is extended to all formulas. If A is atomic

$$\begin{aligned} \Delta |=_{\text{S4}} \boldsymbol{M}(A) &\Leftrightarrow \Delta |=_{\text{S4}} \Box A \\ &\Leftrightarrow |=_{\text{S4}} (\forall \Delta^*) \Delta^* |=_{\text{S4}} A \\ &\Leftrightarrow |=_{\text{S4}} (\forall \Delta^*) \Delta^* |=_{\text{I}} A \\ &\Leftrightarrow \Delta |=_{\text{I}} A \end{aligned}$$

and the result follows by lemma 7.2. [44]

***Lemma* 7.4:** Given an S4 counter-model for $\boldsymbol{M}(X)$, there is an intuitionistic counter-model for X.

Proof: We have $<\mathcal{G}, \mathcal{R}, |=_{\text{S4}}>$ an S4 model such that for some $\Gamma \in \mathcal{G}\, \Gamma =|_{\text{S4}} \boldsymbol{M}(X)$. We take for our intuitionistic model $<\mathcal{G}, \mathcal{R}, |=_{\text{I}}>$ where $|=_{\text{I}}$ is defined by

$$\Delta |=_{\text{I}} A \text{ if } \Delta |=_{\text{S4}} \boldsymbol{M}(A)$$

for A atomic and any Δ in $\mathcal{G}$, and $|=_{\text{I}}$ is extended to all formulas. Now the result follows by lemma 7.2. [45]

FIRST ORDER INTUITIONISTIC LOGIC

SEMANTICS

§ 1. Formulas

We begin with the following:
(1). denumerably many individual variables *x, y, z, w*, …
(2). denumerably many individual parameters *a, b, c, d*, …
(3). for each positive integer *n*, a denumerable list of *n*-ary predicates A^n, B^n, C^n, D^n, …
(4). connectives, quantifiers, parentheses, ∧, ∨, →, ~, ∃, ∀, (,).

An *atomic formula* is an *n*-ary predicate symbol A^n followed by an *n*-tuple of individual symbols (variables or parameters), thus $A^n(\alpha_1, \ldots, \alpha_n)$. A formula is anything resulting from the following recursive rules:

F0. Any atomic formula is a formula.
Fl. If *X* is a formula, so is ~*X*.
F2, 3, 4. If *X* and *Y* are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$.
F5, 6. If *X* is a formula and *x* is a variable, $(\forall x)X$ and $(\exists x)X$ are formulas.

Subformulas and the *degree of a* formula are defined as usual. The property of uniqueness of composition of a *formula* still holds. We note the usual properties of substitution, and we use the following notation: If *X* is a formula and α and β are individual symbols, by $X(^{\alpha}{}_{\beta})$ we mean the result of substituting β for every occurrence of α in *X* (every free occurrence in case α is a variable). We usually denote this informally as follows: we write *X* as $X(\alpha)$ and $X(^{\alpha}{}_{\beta})$ as $X(\beta)$. It will be clear from the 46 context what is meant. We again use parentheses in an informal manner and we omit superscripts on predicates.

Although the definition of formula as stated allows unbound occurrences of variables in formulas, we shall assume, unless otherwise stated, that all variables in a formula are bound. Notation like $X(x)$ however, indicates that x may have free occurrences in *X*.

§ 2. Models an validity

In this section we define the notion of a first order intuitionistic model, and first order intuitionistic validity, referred to respectively as model and validity. This modeling structure is due to Kripke and may be found, in different notation, in [13] (see also [18]). The notions of ch. 1, if needed, will be referred to as propositional notions to distinguish them.

If $\mathcal{P}$ is a map from to sets of parameters, by $\mathcal{P}(\overline{\Gamma})$ we mean the set of all formulas which may be constructed using only parameters of $\mathcal{P}(\Gamma)$. By a (*first order intuitionistic*) *model* we mean an ordered quadruple $<\mathcal{G}, \mathcal{R}, |\!=, \mathcal{P}>$, where $\mathcal{G}$ is a non-empty set, $\mathcal{R}$ is a transitive, reflexive relation on $\mathcal{G}$, $|\!=$ is a relation between elements of $\mathcal{G}$ and formulas, and $\mathcal{P}$ is a map from $\mathcal{G}$ to non-empty sets of parameters, satisfying the following conditions:

for any $\Gamma \in G$

Q0. $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma^*)$,
Q1. $\Gamma \models A \Rightarrow A \in \mathcal{P}(\Gamma)$ for A atomic,
Q2. $\Gamma \models A \Rightarrow \Gamma \models A$ for A atomic,
Q3. $\Gamma \models (X \wedge Y) \Leftrightarrow \Gamma \models X$ and $\Gamma \models Y$,
Q4. $\Gamma \models (X \vee Y) \Leftrightarrow (X \vee Y) \in \mathcal{P}(\Gamma)$ and $\Gamma \models X$ or $\Gamma \models Y$,
Q5. $\Gamma \models {\sim}X \Leftrightarrow {\sim}X \in \mathcal{P}(\Gamma)$ and for all Γ^*, $\Gamma^* =\mid X$,
Q6. $\Gamma \models (X \rightarrow Y) \Leftrightarrow (X \rightarrow Y) \in \mathcal{P}(\Gamma)$ and for all Γ^*, if $\Gamma^* \models X$, then $\Gamma^* \models Y$,
Q7. $\Gamma \models (\exists x)X(x)$ for some $a \in \mathcal{P}(\Gamma)$ $\Gamma \models X(a)$,
Q8. $\Gamma \models (\forall x)X(x)$ for every Γ^* and for every $a \in \mathcal{P}(\Gamma^*)$ $\Gamma^* \models X(a)$.

We call a particular formula X *valid in the model* $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ if for all $\Gamma \in G$ such that $X \in \mathcal{P}(\Gamma)$ $\Gamma \models X$. X is called *valid* if X is valid in all models.

§ 3. Motivation

The intuitive interpretation given in ch. 1 § 3 for the propositional case may be extended to this first order situation. [47]

In one's usual mathematical work, parameters may be introduced as one proceeds, but having introduced a parameter, of course it remains introduced. This is what the map $\mathcal{P}$ is intended to represent. That is, for $\Gamma \in \mathcal{G}$ Γ is a state of knowledge, and $\mathcal{P}(\Gamma)$ is the set of all parameters introduced to reach Γ. (Or in a stricter intuitive sense, $\mathcal{P}(\Gamma)$ is the set of all mathematical entities constructed by time Γ). Since parameters, once introduced, do not disappear, we have Q0. Q2-6 are as in the propositional case. Q7 should be obvious. Q8 may be explained: to know $(\forall x)X(x)$ at Γ, it is not enough merely to know $X(a)$ for every parameter a introduced so far (i.e. for all $a \in \mathcal{P}(\Gamma)$). Rather one must know $X(a)$ for all parameters which can ever be introduced (i.e. for all $a \in \mathcal{P}(\Gamma^*)$ $\Gamma^* \models X(a)$).

The restrictions Q1, and in Q4, Q5 and Q6 are simply to the effect that it makes no sense to say we know the truth of a formula X if X uses parameters we have not yet introduced. It would of course make sense to add corresponding restrictions to Q3, Q7 and Q8, but it is not necessary. The original explanation of Kripke may be found in [13]. For a different but related model theory in terms of forcing see [5].

§ 4. Some properties of models

***Theorem* 4.1:** In any model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$, for any $\Gamma \in \mathcal{G}$, if $\Gamma \models X$, then $X \in \mathcal{P}(\Gamma)$.
Proof: A straightforward induction on the degree of X.

***Theorem* 4.2:** In any model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$, for any formula X, if $\Gamma \models X$, then $\Gamma^* \models X$.
Proof: Also a straightforward induction on the degree of X.

***Theorem* 4.3:** Let $\mathcal{G}$ be a non-empty set, $\mathcal{R}$ be a transitive reflexive relation on $\mathcal{G}$, and $\mathcal{P}$ be a map from $\mathcal{G}$ to non-empty sets of parameters such that $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma^*)$ for all $\Gamma \in \mathcal{G}$. Suppose $\models$ is a relation between elements of G and atomic formulas such that $\Gamma \models A \Rightarrow A \in \mathcal{P}(\Gamma)$. Then $\Gamma \models$ can be extended in one and only one way to a relation, also denoted by $\models$, between and formulas, such that $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is a model.
Proof: A straightforward extension of the corresponding propositional proof.

Definition 4.4: Let $\langle\mathcal{G}, \mathcal{R}, \models, \mathcal{P}\rangle$ be a model and suppose a is some [48] parameter such that $a \notin \cup_{\Gamma\in\mathcal{G}} \mathcal{P}(\Gamma)$. By $\langle\mathcal{G}, \mathcal{R}, \models, \mathcal{P}\rangle(^{b}{}_{a})$ we mean the model $\langle\mathcal{G}, \mathcal{R}, \models', \mathcal{P}\rangle$ defined as follows:

$\mathcal{P}'(\Gamma)$ is the same as $\mathcal{P}(\Gamma)$ except for containing a in place of b if $\mathcal{P}(\Gamma)$ contains b.

For A atomic $\Gamma \models A \Rightarrow \Gamma \models' A(^{b}{}_{a})$, and $\models'$ is extended to all formulas.

***Lemma* 4.5:** Let $\langle G, R, \models, P\rangle$ be a model, $a \notin \cup_{\Gamma\in G} P(\Gamma)$, $\langle\mathcal{G}, \mathcal{R}, \models', \mathcal{P}'\rangle$ be $\langle G, R, \models, P\rangle(^{b}{}_{a})$. Then for any formula X not containing a

$$\Gamma \models X \Rightarrow \Gamma \models' X(^{b}{}_{a}).$$

Proof: By an easy induction on the degree of X.

Definition 4.6: Let $\langle\mathcal{G}, \mathcal{R}, \models, \mathcal{P}\rangle$ be a model and suppose a is some parameter such that $a \notin \cup_{\Gamma\in\mathcal{G}} \mathcal{P}(\Gamma)$. By $\langle\mathcal{G}, \mathcal{R}, \models, \mathcal{P}\rangle_{b=a}$ we mean the model $\langle\mathcal{G}, \mathcal{R}, \models', \mathcal{P}'\rangle$ defined as follows:

$\mathcal{P}'(\Gamma)$ is the same as $\mathcal{P}(\Gamma)$ except for containing a as well as b whenever $\mathcal{P}(\Gamma)$ contains b.

For A atomic $\Gamma \models A \Rightarrow \Gamma \models' A'$, where A' is like A except for containing a at zero or more places where A contains b, and $\models'$ is extended to all formulas.

***Lemma* 4.7:** Let $\langle G, R, \models, P\rangle$ be a model $a \notin \cup_{\Gamma\in G} P(\Gamma)$, and let $\langle G, R, \models', P'\rangle$ be $\langle G, R, \models, P\rangle_{b=a}$. Then if X is any formula not containing a, and if X' is like X except for containing a at zero or more places where X contains b

$$\Gamma \models X \Rightarrow \Gamma \models' X'.$$

Proof: Again an easy induction on the degree of X.

§ 5. Examples

We show that two theorems of classical logic are not intuitionistically valid:

(1). $\vdash_{C} \sim\sim(\forall x)(A(x) \vee \sim A(x))$,

but the following is an intuitionistic counter-model for it. We take the natural numbers as parameters. Let

$$\mathcal{G} = \{\Gamma_i \mid i = 0, 1, 2, ...\},$$
$$\Gamma_i \mathcal{R} \Gamma_j \text{ iff } i \leq j,$$
$$\mathcal{P}(\Gamma_i) = \{\mathbf{1, 2, ...,}\ \boldsymbol{i}, \boldsymbol{i} + \mathbf{1}\}$$ [49]

$\Gamma_n \models A(\boldsymbol{i})$ iff $i \leq n$ and $\models$ is extended to all formulas. We may give this model schematically by

$\Gamma_0\,[\mathbf{1}]$
|
$\Gamma_1\,[\mathbf{1, 2}] \models A(\mathbf{1})$
|
$\Gamma_2\,[\mathbf{1, 2, 3}] \models A(\mathbf{1}), A(\mathbf{2})$
|
$\Gamma_3\,[\mathbf{1, 2, 3, 4}\,] \models A(\mathbf{1}), A(\mathbf{2}), A(\mathbf{3})$
|
…

We claim no $\Gamma_i \models \sim\sim(\forall x)(A(x) \vee \sim A(x))$. Suppose instead that

$$\Gamma_i \models \sim\sim(\forall x)(A(x) \vee \sim A(x)).$$

Then for some $j \geq i$

$$\Gamma_j \models (\forall x)(A(x) \vee \sim A(x)).$$

But $\boldsymbol{j}\mathbf{+1} \in \mathcal{P}(\Gamma_i)$, so

$$\Gamma_j \models A(\boldsymbol{j}\mathbf{+1}) \vee \sim A(\boldsymbol{j}\mathbf{+1}).$$

But $\Gamma_j =\!|\, A(\boldsymbol{j}\mathbf{+1})$ since $j+1 > j$, and if $\Gamma_j \models \sim A(\boldsymbol{j}\mathbf{+1})$, then since $\Gamma_j \mathcal{R} \Gamma_{j+1}$, $\Gamma_{j+1} =\!|\, A(\boldsymbol{j}\mathbf{+1})$, a contradiction.

(2). $\vdash_C (\forall x)(A \vee B(x)) \rightarrow (A \vee (\forall x)B(x))$,

but an intuitionistic counter-model is the following, where parameters are again integers:

$$\mathcal{G} = \{\Gamma_1, \Gamma_2\},$$
$$\Gamma_1 \mathcal{R} \Gamma_2,\ \Gamma_1 \mathcal{R} \Gamma_1,\ \Gamma_2 \mathcal{R} \Gamma_2,$$
$$\mathcal{P}(\Gamma_1) = \{\mathbf{1}\},\ \mathcal{P}(\Gamma_2) = \{\mathbf{1, 2}\},$$
$$\Gamma_1 \models B(\mathbf{1}),\ \Gamma_2 \models B(\mathbf{1}),\ \Gamma_2 \models A,$$

and $\models$ is extended to all formulas. Schematically, this is

$\Gamma_1\,[\mathbf{1}] \models B(\mathbf{1})$
|
$\Gamma_2\,[\mathbf{1, 2}] \models B(\mathbf{1}), A$

To show this is a counter-model, first we claim

$$\Gamma_1 \models (\forall x)(A \vee B(x)).$$ [50]

This follows because $\Gamma_1 \models B(\mathbf{1})$. Hence

$$\Gamma_1 \models A \vee B(\mathbf{1})$$

and $\Gamma_2 \models A$, so

$$\Gamma_2 \models A \vee B(\mathbf{1}) \text{ and } \Gamma_2 \models A \vee B(\mathbf{2}).$$

But $\Gamma_1 =\!| A$ and moreover $\Gamma_1 =\!| (\forall x)B(x)$ since $\Gamma_2 =\!| B(\mathbf{2})$. Thus $\Gamma_1 =\!| A \vee (\forall x)B(x)$.

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Commento:

§ 6. Truth and almost-truth sets

In classical first order logic, a set $\mathcal{S}$ of formulas is sometimes called a *truth set* if

(1). $X \wedge Y \in \mathcal{S} \Leftrightarrow X \in \mathcal{S}$ and $Y \in \mathcal{S}$,
(2). $X \vee Y \in \mathcal{S} \Leftrightarrow X \in \mathcal{S}$ or $Y \in \mathcal{S}$,
(3). $\sim X \in \mathcal{S} \Leftrightarrow X \notin \mathcal{S}$,
(4). $X \rightarrow Y \in \mathcal{S} \Leftrightarrow X \notin \mathcal{S}$ or $Y \in \mathcal{S}$,
(5). $(\exists x)X(x) \in \mathcal{S} \Leftrightarrow X(a) \in \mathcal{S}$ for some parameter a,
(6). $(\forall x)X(x) \in \mathcal{S} \Leftrightarrow X(a) \in \mathcal{S}$ for every parameter a,

where there is some fixed set of parameters, X and Y are formulas involving only these parameters, and (5) and (6) refer to this set of parameters.
We now call $\mathcal{S}$ an almost-truth set if it satisfies (l)-(5) above and

(6a). $(\forall x)X(x) \in \mathcal{S} \Rightarrow X(a) \in \mathcal{S}$ for every parameter a.

It is one form of the classical completeness theorem that for any pure (i.e. with no parameters) formula X, X is a classical theorem if and only if X is in every truth set.
We leave the reader to show

***Theorem* 6.1:** If X is pure and contains no occurrence of the universal quantifier, X is in every truth set if and only if X is in every almost-truth set.

§ 7. Complete sequences

The method used in this section was adapted from forcing techniques, and is due to Cohen [3]. [51]

Definition 7.1: In the model $< \mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$, we call $\mathcal{C} \subseteq \mathcal{R}$ an $\mathcal{R}$*-chain* if

$$\Gamma, \Delta \in \mathcal{C} \Rightarrow \Gamma \mathcal{R} \Delta \text{ or } \Delta \mathcal{R} \Gamma.$$

If $\mathcal{C}$ is an $\mathcal{R}$-chain, by $\mathcal{C}'$ we mean $\{X \mid \text{ for some } \Gamma \in \mathcal{C}, \Gamma \models X\}$.
If $\mathcal{C}$ is an $\mathcal{R}$-chain, $\mathcal{C}$ is called *complete* if, for every formula X with parameters used in $\mathcal{C}'$, $X \vee \sim X \in \mathcal{C}'$.

***Lemma* 7.2**: Let $\mathcal{C}$ be a complete $\mathcal{R}$-chain in the model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$. Then $\mathcal{C}'$ is an almost-truth set.

Proof: This is a straightforward verification of the cases. We give case (4) as an illustration.

Suppose $(X \rightarrow Y) \in \mathcal{C}'$. Then for some $\Gamma \in \mathcal{C}\ \Gamma \models X \rightarrow Y$. Now either $X \notin \mathcal{C}'$ or $X \in \mathcal{C}'$. If $X \in \mathcal{C}'$, then for some $\Delta \in \mathcal{C}\ \Delta \models X$. Let Ω be the $\mathcal{R}$-last of Γ and Δ. Then $\Omega \models X$ and $\Omega \models X \rightarrow Y$, so $\Omega \models Y$ and $Y \in \mathcal{C}'$. Thus $X \notin \mathcal{C}'$ or $Y \in \mathcal{C}'$.

Conversely suppose $(X \rightarrow Y) \notin \mathcal{C}'$. Then $\sim X \notin \mathcal{C}'$, since $\mathcal{C}'$ is closed under *modus ponens* and contains $\sim X \rightarrow (X \rightarrow Y)$ as is easily shown. But $X \vee \sim X \in \mathcal{C}'$, hence $X \in \mathcal{C}'$. Further $Y \in \mathcal{C}'$, since again $Y \rightarrow (X \rightarrow Y) \in \mathcal{C}'$.

***Lemma* 7.3**: Let $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ be a model, $\Gamma \in \mathcal{G}$ and $X \in \mathcal{P}(\Gamma)$. There is some $\Gamma^* \in \mathcal{G}$ such that $\Gamma^* \models X \vee \sim X$.

Proof: Either some $\Gamma^* \models X$ and we are done, or no $\Gamma^* \models X$ in which case $\Gamma \models \sim X$ and we are done.

***Theorem* 7.4:** Let $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ be a model and $\Gamma \in \mathcal{G}$. Then Γ can be included in some complete $\mathcal{R}$-chain $\mathcal{C}$ such that $\mathcal{C}'$ is an almost-truth set.

Proof: There are only countably many formulas, $X_1, X_2, X_3, \ldots$ We define a countable $\mathcal{R}$-chain $\{\Gamma_0, \Gamma_1, \Gamma_02, \ldots\}$ as follows:

Let Γ_0 be Γ.

Having defined Γ_n, if $X_{n+1} \notin \mathcal{P}(\Gamma^*)$ for any Γ_n^*, let Γ_{n+1} be Γ_n. If $X_{n+1} \in \mathcal{P}(\Gamma_n^*)$ for some Γ_n^*, then Γ_n^*, by lemma 7.3, has an $\mathcal{R}$-successor Γ_n^{**} such that $\Gamma_n^{**} \models X_{n+1} \vee \sim X_{n+1}$. Let Γ_{n+1} be this Γ_n^{**}.

Let $\mathcal{C}$ be $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$. Clearly $\mathcal{C}$ is complete, and by lemma 7.2 $\mathcal{C}'$ is an almost-truth set.

§ 8. A connection with classical logic

The first theorem of this section is essentially theorem 59(b) of [10 p. 492], but there it is proved prooftheoretically and here semantically. [52]

***Theorem* 8.1:** Let X be a pure formula. If X is in every classical almost-truth set, $\sim\sim X$ is intuitionistically valid.

Proof: Suppose $\sim\sim X$ is not valid. Then there is a model $< \mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and a $\Gamma \in \mathcal{G}$ such that $\Gamma \,=\!|\, \sim\sim X$. Then for some $\Gamma^* \in \mathcal{G}\ \Gamma^* \models \sim X$. Now Γ^* can, by theorem 7.4, be included in an $\mathcal{R}$-chain $\mathcal{C}$ such that $\mathcal{C}'$ is an almost-truth set. But $\sim X \in \mathcal{C}'$, so that $X \notin \mathcal{C}'$.

***Theorem* 8.2:** If X is intuitionistically valid, then X is classically valid (for X pure).

Proof: As before, if X is not classically valid, there is a truth set $\mathcal{S}$ not containing X. But it is easily shown that if $\mathcal{G} = \{\mathcal{S}\}$, $\mathcal{S}\mathcal{R}\mathcal{S}$, $\mathcal{S} \models Y$ if $Y \in \mathcal{S}$, and $\mathcal{P}(\mathcal{S})$ is the set of all parameters occurring in $\mathcal{S}$, the resulting $< \mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is a model in which X is not valid.

***Theorem* 8.3:** If X is a pure formula with no occurrence of the universal quantifier, then X is classically valid if and only if $\sim\sim X$ is intuitionistically valid.

Proof:

$$\sim\sim X \text{ intuitionistically valid} \Rightarrow \sim\sim X \text{ classically valid}$$
$$\Rightarrow X \text{ classically valid.}$$

Conversely

$$X \text{ classically valid} \Rightarrow X \text{ is in every truth set}$$
$$\Rightarrow X \text{ is in every almost-truth set}$$
$$\Rightarrow \sim\sim X \text{ is intuitionistically valid.}$$

Remark 8.4: This result [due to Kolmogorov, 1925] will be of fundamental importance in part II.

***Corollary* 8.5:** First order intuitionist logic is undecidable.

Proof: Classical first order logic is undecidable, and every classical formula is classically equivalent to a formula with no universal quantifiers.

Remark 8.6: That theorem 8.3 cannot be extended to all formulas is shown by example (1) in § 5.[53]

FIRST ORDER INTUITIONISTIC LOGIC

PROOF THEORY

§ 1. Beth tableaus

The following is an extension of the system of ch. 2 § 1 to the first order case (see [2]). Everything is as it was there, except that four reduction rules are added to the list. These are

$$T\exists: \quad \frac{\boldsymbol{S},\ T(\exists x)X(x)}{\boldsymbol{S},\ TX(a)} \text{ provided } a \text{ is new} \qquad F\exists: \quad \frac{\boldsymbol{S},\ F\exists(x)X(x)}{\boldsymbol{S},\ FX(a)}$$

$$T\forall: \quad \frac{\boldsymbol{S},\ T(\forall x)X(a)}{\boldsymbol{S},\ TX(a)} \qquad F\forall: \quad \frac{\boldsymbol{S},\ F(\forall x)X(a)}{\boldsymbol{S}_T,\ FX(a)} \text{ provided } a \text{ is new}$$

(Note the $\boldsymbol{S}_T$ in rule $F\forall$.) In rules $F\exists$ and $T\forall$, a may be any parameter whatsoever. In rules $T\exists$ and $F\forall$, the parameter a introduced must not occur in any formula of $\boldsymbol{S}$ or in the formula $X(x)$.

The corresponding classical tableau system is like the above, but in rule $F\forall$ $\boldsymbol{S}_T$ is replaced by $\boldsymbol{S}$. As in ch. 2 § 1 interpretations differ. Classically the interpretation is as it was in the propositional case. The restrictions on parameters in $T\exists$ and $F\forall$ are for obvious reasons. In the intuitionistic system the difference between $T\exists$ and $F\forall$ may be explained [54] as follows. Suppose we have proved $(\exists x)X(x)$. Since (intuitionistically) the only existence proofs are constructive, there must already be an instance $X(a)$ which we have proved. Thus rule $T\exists$. But suppose we have not proved $(\forall x)X(x)$. We might have proved all instances so far encountered, but it must be possible (i.e. compatible with our present knowledge) that we will at some time encounter an instance for which we will have no proof. However, this might happen at some time in the future, by which time we may have proved some things we do not know (some $FZ \in \boldsymbol{S}$ might become TZ). Hence the restriction to $\boldsymbol{S}_T$ in rule $F\forall$.

As in the propositional case, we proceed to show correctness and completeness (in two ways) of this system.

The following two examples illustrate proofs in the system:

(1). $\vdash_I (\forall x)X(x) \rightarrow \sim(\exists x)\sim X(x)$.

The proof is

$\{\{F(\forall x)X(x) \rightarrow \sim(\exists x)\sim X(x)\}\}$,
$\{\{T(\forall x)X(x),\ F\sim(\exists x)\sim X(x)\}\}$,
$\{\{T(\forall x)X(x),\ T(\exists x)\sim X(x)\}\}$,
$\{\{T(\forall x)X(x),\ T\sim X(a)\}\}$,

$\{\{TX(a), T{\sim}X(a)\}\}$,
$\{\{TX(a), FX(a)\}\}$.

(2). $\vdash_I {\sim}(\exists x) \sim [X(x) \to Y(x)] \to (\forall x)[\sim Y(x) \to \sim X(x)]$.

The proof is

$\{\{F{\sim}(\exists x) \sim [X(x) \to Y(x)] \to (\forall x)[\sim Y(x) \to \sim X(x)]\}\}$,
$\{\{T{\sim}(\exists x) \sim [X(x) \to Y(x)], F(\forall x)[\sim Y(x) \to \sim X(x)]\}\}$,
$\{\{T{\sim}(\exists x) \sim [X(x) \to Y(x)], F[\sim Y(a) \to \sim X(a)]\}\}$,
$\{\{T{\sim}(\exists x) \sim [X(x) \to Y(x)], T{\sim}Y(a), F{\sim}X(a)]\}\}$,
$\{\{T{\sim}(\exists x) \sim [X(x) \to Y(x)], T{\sim}Y(a), TX(a)]\}\}$,
$\{\{F(\exists x) \sim [X(x) \to Y(x)], T{\sim}Y(a), TX(a)]\}\}$,
$\{\{F \sim [X(a) \to Y(a)], T{\sim}Y(a), TX(a)]\}\}$,
$\{\{T[X(a) \to Y(a)], T{\sim}Y(a), TX(a)]\}\}$,
$\{\{FX(a), T{\sim}Y(a), TX(a)]\}\}$, $\{\{TY(a), T{\sim}Y(a), TX(a)]\}\}$,
$\{\{FX(a), T{\sim}Y(a), TX(a)]\}\}$, $\{\{TY(a), FY(a), TX(a)]\}\}$,

§ 2. Correctness of Beth tableaus

Definition 2.1: Let $\boldsymbol{S} = \{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$ be a set of signed [55] formulas, $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ a model, and $\Gamma \in \mathcal{G}$. We say Γ *realizes* $\boldsymbol{S}$ if $X_i \in \mathcal{P}(\Gamma)$, $Y_i \in \mathcal{P}(\Gamma)$, and $\Gamma \models X_i$, $\Gamma =\!| \ Y_j$ $(i = 1, \ldots, n, j = 1, \ldots, m)$. A set S is *realizable* if something realizes it.

A configuration $\mathcal{C}$ is *realizable* if one of its elements is realizable.

***Lemma* 2.2:** Let Q stand for either the sign T or the sign F. If $\boldsymbol{S}$, $QX(b)$ is realizable and if a is a parameter which does not occur in $\boldsymbol{S}$ or in X (so $a \neq b$) then $\boldsymbol{S}$, $QX(a)$ is realizable.

Proof: Suppose in the model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ Γ realizes $\boldsymbol{S}$, $QX(b)$. Choose a new parameter $c \notin \cup_{\Gamma \in \mathcal{G}} \mathcal{P}(\Gamma)$ (we can always construct a new parameter). Let $<\mathcal{G}, \mathcal{R}, \models', \mathcal{P}'>$ be $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>(^a{}_c)$ (see ch. 4 § 4). Since a does not occur in $\boldsymbol{S}$ or X, by lemma 4.4.5, in this new model Γ realizes $\boldsymbol{S}$, $QX(b)$. But now $a \notin \cup_{\Gamma \in \mathcal{G}} \mathcal{P}'(\Gamma)$, so we may define a third model $<\mathcal{G}, \mathcal{R}, \models'', \mathcal{P}''>$ as $<\mathcal{G}, \mathcal{R}, \models', \mathcal{P}'>_{b=a}$. By lemma 4.4.7 in this third model Γ realizes $\boldsymbol{S}$, $QX(a)$.

***Lemma* 2.3:** If $\boldsymbol{S}$, $T(\exists x)X(x)$ is realizable, and if a does not occur in $\boldsymbol{S}$ or $X(x)$, then $\boldsymbol{S}$, $TX(a)$ is realizable.

Proof: Suppose in the model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ Γ realizes $\boldsymbol{S}$, $T(\exists x)X(x)$. Then $\Gamma \models (\exists x)X(x)$, so for some $b \in \mathcal{P}(\Gamma)$ $\Gamma \models X(b)$. Thus Γ realizes S, $TX(b)$. If $a = b$ we are done. If not, by lemma 2.2 we are done.

***Lemma* 2.4:** If $\boldsymbol{S}$, $F(\exists x)X(x)$ is realizable and if a is any parameter, $\boldsymbol{S}$, $FX(a)$ is realizable.

Proof: Suppose in the model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ Γ realizes $\boldsymbol{S}$, $F(\exists x)X(x)$. Then $\Gamma =\!| \ (\exists x)X(x)$. If $a \in \mathcal{P}(\Gamma)$, $\Gamma =\!| \ X(a)$ and we are done. If $a \notin \mathcal{P}(\Gamma)$, a cannot occur in $\boldsymbol{S}$ or X by the definition of realizability. But $\mathcal{P}(\Gamma) \neq \varnothing$, so there is a $b \in \mathcal{P}(\Gamma)$ with $b \neq a$ and $\Gamma =\!| \ X(b)$. Thus $\boldsymbol{S}$, $FX(b)$ is realizable. Now use lemma 2.2.

***Lemma* 2.5:** If $\boldsymbol{S}$, $T(\forall x)X(x)$ is realizable and if a is any parameter, $\boldsymbol{S}$, $TX(a)$ is realizable.

Proof: Similar to that of lemma 2.4.

***Lemma* 2.6:** If $\boldsymbol{S}$, $F(\forall x)X(x)$ is realizable and if a is any parameter which does not occur in $\boldsymbol{S}$ or $X(x)$, then $\boldsymbol{S}_T$, $FX(a)$ is realizable.

Proof: Suppose in the model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ Γ realizes $\boldsymbol{S}$, $F(\forall x)X(x)$. Then $\Gamma =\!|$ $(\forall x)X(x)$. But $X(x) \in \mathcal{P}(\Gamma)$, so there is a Γ^* such that $\Gamma^* =\!|$ $X(b)$ for some $b \in \mathcal{P}(\Gamma^*)$. Of course Γ^* realizes $\boldsymbol{S}_T$. If $b = a$ we are done. If not, since $\boldsymbol{S}_T$, $X(b)$ is realizable, by lemma 2.2 we are done. [56]

***Theorem* 2.7:** Let $C_1, C_2, \ldots, C_n$ be a tableau. If C_i is realizable, so is C_{i+1}.

Proof: We pass from C_i to C_{i+1} by the application of some reduction rule. All the propositional rules were dealt with in ch. 2. The four new (first order) rules are handled by lemmas 2.3-2.6.

***Corollary* 2.8:** If X is provable, X is valid.

Proof: Exactly as in the propositional situation.

§ 3. Hintikka collections

This section generalizes the definitions of ch. 2 § 3 to the first order setting. Recall that a finite set of signed formulas is consistent if no tableau for it is closed. We say an infinite set is consistent if every finite subset is.

Let $\mathcal{G}$ be a collection of sets of signed formulas. If $\Gamma \in \mathcal{G}$, by $\mathcal{P}(\Gamma)$ we mean the collection of all parameters occurring in formulas in Γ. If $\Gamma, \Delta \in \mathcal{G}$, by $\Gamma\mathcal{R}\Delta$ we mean $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$ and $\Gamma_T \subseteq \Delta$.

Definition 3.1: We call $\mathcal{G}$ a (first *order*) *Hintikka collection* if, for any $\Gamma \in \mathcal{G}$ Γ is consistent and

$TX \wedge Y \in \Gamma \Rightarrow TX \in \Gamma$ and $TY \in \Gamma$,
$FX \vee Y \in \Gamma \Rightarrow FX \in \Gamma$ and $FY \in \Gamma$,
$TX \vee Y \in \Gamma \Rightarrow TX \in \Gamma$ or $TY \in \Gamma$,
$FX \wedge Y \in \Gamma \Rightarrow FX \in \Gamma$ or $FY \in \Gamma$,
$T{\sim}X \in \Gamma \Rightarrow FX \in \Gamma$
$TX \rightarrow Y \in \Gamma \Rightarrow FX \in \Gamma$ or $TY \in \Gamma$,
$F{\sim}X \in \Gamma \Rightarrow$ for some $\Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$, $TX \in \Delta$,
$FX \rightarrow Y \in \Gamma \Rightarrow$ for some $\Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$, $TX \in \Delta$, $FY \in \Delta$,
$T(\forall x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for all $a \in \mathcal{P}(\Gamma)$,
$F(\exists x)X(x) \in \Gamma \Rightarrow FX(a) \in \Gamma$ for all $a \in \mathcal{P}(\Gamma)$,
$T(\exists x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for all $a \in \mathcal{P}(\Gamma)$,
$F(\forall x)X(x) \in \Gamma \Rightarrow$ for some $\Delta \in \mathcal{G}$ $\Gamma\mathcal{R}\Delta$ and
for some $a \in \mathcal{P}(\Delta)$ $TX(a) \in \Delta$.

Definition 3.2: If $\mathcal{G}$ is a Hintikka collection, we call $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ a *model* for $\mathcal{G}$ if

(1). $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is a model, [57]
(2). $\mathcal{P}$ and $\mathcal{R}$ are as above,

(3). for all $\Gamma \in \mathcal{G}\ TX \in \Gamma \Rightarrow \Gamma \models X$ and $FX \in \Gamma \Rightarrow \Gamma =\!\mid X$.

***Theorem* 3.3:** There is a model for any Hintikka collection.

Proof: Suppose we have a Hintikka collection $\mathcal{G}$, $\mathcal{P}$ and $\mathcal{R}$ are as defined above. If A is atomic, let $\Gamma \models A$ if $TA \in \Gamma$, and extend to all formulas. The result $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is a model. We claim it is a model for $\mathcal{G}$. We show property (3) by induction on the degree of X.

The propositional cases were done in ch. 2 § 3. Of the four new cases we only do two as an illustration.

Suppose the result known for all subformulas of the formula in question. Then

$$
\begin{aligned}
T(\forall x)X(x) \in \Gamma &\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \Rightarrow T(\forall x)X(x) \in \Delta) \\
&\qquad \text{(since } \Gamma_T \subseteq \Delta \text{ if } \Gamma \mathcal{R} \Delta) \\
&\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \Rightarrow ((\forall a \in P(\Delta))TX(a) \in \Delta)) \\
&\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \Rightarrow ((\forall a \in P(\Delta))\ \Delta \models X(a))) \\
&\Rightarrow \Gamma \models (\forall x)\ X(x).
\end{aligned}
$$

Conversely

$$
\begin{aligned}
F(\forall x)X(x) \in \Gamma &\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \text{ and } (\exists a \in \mathcal{P}(\Delta))(FX(a) \in \Delta)) \\
&\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \text{ and } (\exists a \in \mathcal{P}(\Delta))(\Delta =\!\mid X(a))) \\
&\Rightarrow \Gamma \models (\forall x)\ X(x).
\end{aligned}
$$

Thus, as in the propositional case, to establish the completeness of Beth tableaus we need only show that if X is not provable, there is a Hintikka collection and a $\Gamma \in \mathcal{G}$ such that $FX \in \Gamma$.

§ 4. Hintikka elements

Definition 4.1: Let Γ be a set of signed formulas and $\boldsymbol{P}$ a set of parameters. We call Γ a *Hintikka element with respect to* $\boldsymbol{P}$ if Γ is consistent and

$TX \wedge Y \in \Gamma \Rightarrow TX \in \Gamma$ and $TY \in \Gamma$,
$FX \vee Y \in \Gamma \Rightarrow FX \in \Gamma$ and $FY \in \Gamma$,
$TX \vee Y \in \Gamma \Rightarrow TX \in \Gamma$ or $TY \in \Gamma$,
$FX \wedge Y \in \Gamma \Rightarrow FX \in \Gamma$ or $FY \in \Gamma$,
$T \sim X \in \Gamma \Rightarrow FX \in \Gamma$,
$TX \rightarrow Y \in \Gamma \Rightarrow FX \in \Gamma$ or $TY \in \Gamma$, [58]
$T(\forall x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for each $a \in \boldsymbol{P}$,
$F(\exists x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for each $a \in \boldsymbol{P}$,
$T(\exists x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for some $a \in \boldsymbol{P}$,

***Theorem* 4.2:** Let Γ be an at most countable, consistent set of signed formulas. Let $\boldsymbol{S}$ be the set of all parameters occurring in formulas in Γ. Let $a_1, a_2, a_3, \ldots$ be a countable list of parameters not in $\boldsymbol{S}$. Let $\boldsymbol{P} = \boldsymbol{S} \cup \{a_1, a_2, a_3, \ldots\}$. Then Γ can be extended to a Hintikka element with respect to $\boldsymbol{P}$.

Proof: Order the (countable) set of all subformulas of formulas in Γ, using only parameters of $\boldsymbol{P}$: $X_1, X_2, X_3, \ldots$ We define a (double) sequence of sets of signed formulas:

Let $\Gamma_0 = \Gamma$. Suppose we have defined Γ_n which is a consistent extension of Γ_0, using only finitely many of a_1, a_2, a_3, ... Let $\Delta_n^1 = \Gamma_n$. We define $\Delta_n^2, \dots, \Delta_n^{n+1}$ and let $\Gamma_{n+1} = \Delta_n^{n+1}$ We do this as follows:

Suppose we have defined Δ_n^k for some k ($1 \leq k \leq n$). Consider the formula X_k. At most one of TX_k, FX_k can be in Δ_n^k (since it is consistent). If neither is, let $\Delta_n^{k+1} = \Delta_n^k$. If one is in Δ_n^k, we have several cases.

Case (la). X_k is $Y \vee Z$ and $TX_k \in \Delta_n^k$. Then one of Δ_n^k, TY or Δ_n^k, TZ is consistent. Let Δ_n^{k+1} be Δ_n^k, TY if consistent, and Δ_n^k, TZ otherwise.

Case (1 b). X_k is $Y \vee Z$ and $FX_k \in \Delta_n^k$. Then Δ_n^k, FY, FZ is consistent. Let this be Δ_n^{k+1}.

The cases

(2a). $TX \wedge Y$,

(2b). $FX \wedge Y$,

(3). $T \sim X$,

(4). $TX \rightarrow Y$,

are all treated in a similar manner.

Case (5a). X_k *is* $(\exists x)X(x)$ and $TX_k \in \Delta_n^k$. Since Δ_n^k uses only finitely many of a_1, a_2, a_3, ..., let a_i be the first one unused. Let Δ_n^{k+1} be Δ_n^k, $TX(a)$. Since a is new, this must also be consistent.

Case (5b). X_k *is* $(\exists x)X(x)$ and $FX_k \in \Delta_n^k$. Let Δ_n^{k+1} be Δ_n^k together with $FX(a)$ for each $\alpha \in \boldsymbol{S}$, and each $\alpha = a_i$ which has been used so far. Then Δ_n^{k+1} is again consistent.

Case (6). $T(\forall x)X(x)$, is treated as we did case (5b).

Case (7). If the signed formula does not come under one of the above cases let $\Delta_n^{k+1} = \Delta_n^k$. [59]

Thus we have defined a sequence $\Gamma_0, \Gamma_1, \Gamma_2, \dots$, Let $\Pi = \cup\Gamma_n$. We claim Π is a Hintikka collection with respect to $\boldsymbol{P}$. The verification of the properties is straightforward.

§ 5. Completeness of Beth tableaus

Supposing X to be not provable, we give a procedure for constructing a sequence of Hintikka elements.

First we order our countable collection of parameters as follows:

$$\begin{array}{l} \boldsymbol{S}_1\!: a_1^1, a_2^1, a_3^1, \dots \\ \boldsymbol{S}_2\!: a_1^2, a_2^2, a_3^2, \dots \\ \boldsymbol{S}_3\!: a_1^3, a_2^3, a_3^3, \dots \\ \dots\ \dots\ \dots\ \dots\ \dots \end{array}$$

where we have placed all the parameters of X in $\boldsymbol{S}_1$, and let $\boldsymbol{P}_n = \boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \dots \cup \boldsymbol{S}_n$.

For this section only, by an F-formula we mean a signed formula of the form $F{\sim}X$, $FX \rightarrow Y$ or $F(\forall x)X$. We may assume once and for all an ordering of all formulas. Now we proceed:

Step (0). X is not provable, so $\{FX\}$ is consistent. Extend it to a Hintikka element with respect to $\boldsymbol{P}_1$. Call the result Γ_1.

Step (1). Take the first F-formula of Γ_1. If this is $F{\sim}X$, consider Γ_{1T}, TX. This is consistent. Extend it to a Hintikka element with respect to $\boldsymbol{P}_2$, call it Γ_2. If the first F-formula is $FX \rightarrow Y$, extend Γ_{1T}, TX, FY to a Hintikka element with respect to $\boldsymbol{P}_2$, Γ_2. If the first F-formula is $F(\forall x)X(x)$, extend Γ_{1T}, $FX(a_1^2)$ to a Hintikka element with respect to $\boldsymbol{P}_2$, Γ_2. In any event Γ_2 is a consistent Hintikka element with respect to P_2. Now call the first F-element of Γ_1 "used". The result of step (1) is $\{\Gamma_1, \Gamma_2\}$.

Suppose at the end of step (n) we have the sequence $\{\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_{2n}\}$ where each Γ_i is a Hintikka element with respect to $\boldsymbol{P}_i$.

Step (n+1). Take the first "unused" F-formula of Γ_1, proceed as in step (1) depending on whether the formula is $F{\sim}X$, $FX \rightarrow Y$ or $F(\forall x)X$. Produce from Γ_{1T}, TX or Γ_{1T}, TX, FY or Γ_{1T}, $FX(a_1^{2n+1})$ a Hintikka element with respect to $\boldsymbol{P}_{2n+1}$, call it $\boldsymbol{\Gamma}_{2n+1}$, and call the formula in question "used". Repeat the same procedure with the first "unused" F-formula of Γ_2, producing a Hintikka element with respect to $\boldsymbol{P}_{2n+2}$, [60] call it Γ_{2n+2} Continue to Γ_{2n}, producing a Hintikka element with respect to P_{2n+1}, call it Γ_{2n+1}. The result of the (n+l)st step is thus

$$\{\Gamma_1, \Gamma_2, ... \Gamma_{2n+1}\}.$$

Let be $\mathcal{G}$ the collection of all Γ_n generated in the above process. We claim $\mathcal{G}$ is a Hintikka collection.

Each $\Gamma_n \in \mathcal{G}$ is a Hintikka element with respect to $\boldsymbol{P}_n$, so $\mathcal{P}(\Gamma_n)$ is $\boldsymbol{P}_n$. Since Γ_n is a Hintikka element with respect to $\mathcal{P}(\Gamma_n)$, to show $\mathcal{G}$ is a Hintikka collection we have only three properties to show.

Suppose for some $\Gamma_n \in \mathcal{G}$, $F(\forall x)X(x) \in \Gamma_n$. By the above construction there must be some $\Gamma_k \in \mathcal{G}$ such that $\Gamma_{nT} \subseteq \Gamma_k$, $\mathcal{P}(\Gamma_n) \subseteq \mathcal{P}(\Gamma_k)$ and $FX(a) \in \Gamma_k$ for some parameter a. Thus $(\exists \Gamma_k \in \mathcal{G})\Gamma_n \mathcal{R} \Gamma_k$ and $FX(a) \in \Gamma_k$ for some $a \in \mathcal{P}(\Gamma_k)$.

The cases $F{\sim}$ and $F{\rightarrow}$ are similar.

Thus $\mathcal{G}$ is a Hintikka collection and $FX \in \Gamma_1 \in \mathcal{G}$, so our completeness theorem is established. We note that in the Hintikka collection resulting, every formula is a subformula of X. We remark also that the construction of § 4 and of this section could be combined into a single sequence of steps.

This proof is a modification of the original proof of Kripke [13].

§ 6. Second completeness proof for Beth tableaus

The following is a Henkin type proof and serves as a transition to the completeness of the axiom system presented in the next few sections. A proof along the same lines but using unsigned formulas was discovered independently by Thomason [21] and by Aczel [1]. The similarity to the algebraic work of ch. 1 § 6 is also noted.

Recall that a finite set of signed formulas Γ is consistent if no tableau for it is closed. An infinite set is consistent if every finite subset is.

Definition 6.1.: Let $\boldsymbol{P}$ be a set of parameters and Γ a set of signed formulas. We call Γ *maximal consistent with respect to* $\boldsymbol{P}$ if

(1). every signed formula in Γ uses only parameters of $\boldsymbol{P}$,

(2). Γ is consistent,

(3). for every formula X with all its parameters from Γ, either $TX \in \Gamma$ or $FX \in \Gamma$ or both Γ, TX and Γ, FX are inconsistent. [61]

***Lemma* 6.2:** Let Γ be a consistent set of signed formulas, and $\boldsymbol{P}$ be a non-empty set of parameters containing at least every parameter used in Γ. Then Γ can be extended to a set Δ which is maximal consistent with respect to $\boldsymbol{P}$.

Proof: $\boldsymbol{P}$ is countable, so we may enumerate all formulas with parameters from Γ: X_1, X_2, X_3, …

Let $\Delta_0 = \Gamma$. Having defined Δ_n, consider X_{n+1}. If Δ_n, TX_{n+1} is consistent, let it be Δ_{n+1}. If not, but if Δ_n, FX_{n+1} is consistent, let it be Δ_{n+1}. If neither holds, let Δ_{n+1} be Δ_n.

Let $\Delta = \cup\Delta_n$. The conclusion of the lemma is now obvious.

Definition 6.3: Let Γ be a set of signed formulas and $\boldsymbol{P}$ a set of parameters. We call Γ *good with respect to P* if

(1). Γ is *maximal consistent* with respect to $\boldsymbol{P}$,
(2). $T(\exists x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for some $a \in \boldsymbol{P}$.

***Lemma* 6.4:** Let Γ be a consistent set of signed formulas, and $\boldsymbol{S}$ be the set of parameters occurring in Γ. Let $\{a_1, a_2, a_3, ...\}$ be a countable set of distinct parameters not in $\boldsymbol{S}$, and let $\boldsymbol{P} = \boldsymbol{S} \cup \{a_1, a_2, a_3, ...\}$. Then Γ can be extended to a set Δ which is good with respect to P.

Proof: $\boldsymbol{P}$ is countable, order the set of formulas with parameters from $\boldsymbol{P}$: X_1, X_2, X_3, … We proceed as follows:

(1). Let $\Delta_0 = \Gamma$.

(2). Extend Δ_0 to a set Δ_1 maximal consistent with respect to $\boldsymbol{S}$.

(3). Take the first X_i (in the above ordering) of the form $T(\exists x)X(x)$ such that $T(\exists x)X(x) \in \Delta_1$ but for no $\alpha \in \boldsymbol{S}$ is $TX(\alpha) \in \Delta_1$. Let $\Delta_2 = \Delta_1, TX(a_1)$. Since a_1 is "new", Δ_2 is consistent.

(4). Extend Δ_2 to a set Δ_3 maximal consistent with respect to $\boldsymbol{S} \cup \{a_1\}$.

(5). Take the first X_i of the form $T(\exists x)X(x)$ such that $T(\exists x)X(x) \in \Delta_3$ but for no $\alpha \in \boldsymbol{S} \cup \{a_1\}$ is $TX(\alpha) \in \Delta_3$. Let $\Delta_4 = \Delta_3, TX(a_2)$. Again Δ_4 is consistent.

(6). Extend Δ_4 to a set Δ_5 maximal consistent with respect to $\boldsymbol{S} \cup \{a_1, a_2\}$

And so on.

Let $\Delta = \cup\Delta_n$. We claim Δ is good with respect to $\boldsymbol{P}$.

First Δ is consistent since each Δ_n is consistent.

If X has all its parameters in $\boldsymbol{P}$, then for some n all the parameters of X are in $\boldsymbol{S} \cup \{a_1, a_2, ..., a_n\}$. But in step ($2n$) we extend Δ_{2n}. to Δ_{2n+1}, a set maximal consistent with respect to $\boldsymbol{S}$ $\{a_1, a_2, ..., a_n\}$. Thus TX or [62] FX is in Δ_{2n+1} and hence in Δ, or neither can be added consistently. Thus Δ is maximal consistent with respect to $\boldsymbol{P}$.

Finally suppose $T(\exists x)X(x) \in \Delta$. We note that the formula dealt with in step (5) is different from the one dealt with in step (3), and the one dealt with in step (7) is different again. Thus we must eventually reach $T(\exists x)X(x)$, and so for some $\alpha \in \boldsymbol{P}$ $TX(\alpha) \in \Delta$. Hence Δ is good with respect to $\boldsymbol{P}$.

Now let us order our countably many parameters as follows:

$$\begin{array}{l} \boldsymbol{S}_1: a_1^1, a_2^1, a_3^1, \dots \\ \boldsymbol{S}_2: a_1^2, a_2^2, a_3^2, \dots \\ \boldsymbol{S}_3: a_1^3, a_2^3, a_3^3, \dots \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \end{array}$$

and let $\boldsymbol{P}_n = S_1 \cup S_2 \cup \dots \cup S_n$. Let $\mathcal{G}$ be the collection of all sets of signed formulas which are good with respect to some $\boldsymbol{P}$. We claim $\mathcal{G}$ is a Hintikka collection.

Suppose $\Gamma \in \mathcal{G}$. Then Γ is good with respect to some $\boldsymbol{P}_i$, say $\boldsymbol{P}_n$. Then $\mathcal{P}(\Gamma)$ (the collection of all parameters of Γ) is $\boldsymbol{P}_n$.

Suppose T$X \wedge Y \in \Gamma$ but $TX \notin \Gamma$. If Γ, $TX \wedge Y$ is consistent, so is Γ, $TX \wedge Y$, TX, and so Γ is not maximal. Thus $TX \in \Gamma$. Similarly $TY \in \Gamma$. Hence

$$TX \wedge Y \in \Gamma \Rightarrow TX \in \Gamma \text{ and } TY \in \Gamma.$$

Similarly we may show

$FX \vee Y \in \Gamma \Rightarrow FX \in \Gamma$ and $FY \in \Gamma$,
$TX \vee Y \in \Gamma \Rightarrow TX \in \Gamma$ or $TY \in \Gamma$,
$FX \wedge Y \in \Gamma \Rightarrow FX \in \Gamma$ or $FY \in \Gamma$,
$T\sim X \in \Gamma \Rightarrow FX \in \Gamma$,
$TX \rightarrow Y \in \Gamma \Rightarrow FX \in \Gamma$ or $TY \in \Gamma$,
$T(\forall x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for every $a \in \mathcal{P}(\Gamma)$,
$F(\exists x)X(x) \in \Gamma \Rightarrow FX(a) \in \Gamma$ for every $a \in \mathcal{P}(\Gamma)$.

Moreover

$T(\exists x)X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for some $a \in \mathcal{P}(\Gamma)$,

since Γ is good with respect to $\boldsymbol{P}_n$. [63]

Suppose $F{\sim}X \in \Gamma$. Since Γ is consistent, Γ_T, TX is consistent. Extend it to a set Δ which is good with respect to $\boldsymbol{P}_{n+1}$ Then $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$ and $\Gamma_T \subseteq \Delta$, so $\Gamma \mathcal{R} \Delta$ and $TX \in \Gamma$.

Similarly, if $FX \rightarrow Y \in \Gamma$, there is a $\Delta \in \mathcal{G}$e such that $\Gamma \mathcal{R} \Delta$, $TX \in \Delta$ and $FY \in \Delta$.

Finally, if $F(\forall x)X(x) \in \Gamma$, since a_1^{n+1} does not occur in Γ, Γ_T, $FX(a_1^{n+1})$ is consistent. Extend it to a set Δ which is good with respect to $\boldsymbol{P}_{n+1}$. Again $\Gamma \mathcal{R} \Delta$ and $FX(a_1^{n+1}) \in \Delta$ for $a_1^{n+1} \in \mathcal{P}(\Delta)$,

Thus $\mathcal{G}$ is a Hintikka collection.

To complete the proof, suppose X is not provable. Then $\{FX\}$ is consistent. Since it has only finitely many parameters, they must all lie in some $\boldsymbol{P}_n$. Extend $\{FX\}$ to a set Γ good with respect to $\boldsymbol{P}_n$. Then $\Gamma \in \mathcal{G}$ and $FX \in \Gamma$. This establishes completeness.

Remark 6.5: The model resulting from this Hintikka collection is a "universal" model in that it is a counter-model for every non-theorem. This is not the case for the model of § 5.

We will show later that, in a sense, this Hintikka collection is the analog of a classical truth set.

§ 7. An axiom system, $\mathcal{A}_1$

The following system was chosen to give a fairly quick completeness proof. It is very close to the system of [10] p. 82.

Axiom schemas:

1. $X \rightarrow (Y \rightarrow X)$,
2. $(X \rightarrow Y) \rightarrow ((X \rightarrow (Y \rightarrow Z)) \rightarrow (X \rightarrow Z))$,
3. $((X \rightarrow Z) \wedge (Y \rightarrow Z)) \rightarrow ((X \vee Y) \rightarrow Z)$,
4. $(X \wedge Y) \rightarrow X$,

5. $(X \wedge Y) \rightarrow Y$,
6. $X \rightarrow (Y \rightarrow (X \wedge Y))$,
7. $X \rightarrow (X \vee Y)$,
8. $Y \rightarrow (X \vee Y)$,
9. $(X \wedge {\sim}X) \rightarrow Y$,
10. $(X \rightarrow {\sim}X) \rightarrow {\sim}X$,
11. $X(a) \rightarrow (\exists x)X(x)$,
12. $(\forall x)X(x) \rightarrow X(a)$. [64]

Rules:

13. $$\frac{X(a) \rightarrow Y}{(\exists x)X(x) \rightarrow Y},$$

14. $$\frac{Y \rightarrow X(a)}{Y \rightarrow (\forall x)X(x)},$$

15. $$\frac{X,\ X \rightarrow Y}{Y}.$$

In rules 13 and 14 the parameter a must not occur in Y. In a deduction from premises the parameter a must not occur in the premises either. We use the usual notation, if X can be deduced from a finite subset of $\boldsymbol{S}$, we write $\boldsymbol{S} \vdash X$. We use $\vdash X$ for $\varnothing \vdash X$. In the next three sections we establish the correctness and completeness of $\mathcal{A}_1$. We introduce a second system $\mathcal{A}_2$, equivalent to $\mathcal{A}_1$, to aid in showing correctness. For use in showing completeness we need the following three lemmas:

***Lemma* 7.1:** The deduction theorem holds for $\mathcal{A}_1$.
Proof: The standard one (e.g. [10] §§ 21, 22).

***Lemma* 7.2:** $$\frac{\vdash (W \wedge Y) \rightarrow X,\ \vdash (W \wedge Z) \rightarrow X,\ \vdash W \rightarrow (Y \vee Z)}{\vdash W \rightarrow X.}$$

Proof:

(1). $(W \wedge Y) \rightarrow X$	by hypothesis, theorem,
(2). $(W \wedge Z) \rightarrow X$	by hypothesis, theorem,
(3). $W \rightarrow (Y \vee Z)$	by hypothesis, theorem,
(4). W	premise,
(5). $Y \vee Z$	by (3), (4), rule 15,
(6). $W \rightarrow (Y \rightarrow (W \wedge Y))$	axiom 6,
(7). $Y \rightarrow (W \wedge Y)$	by (4), (6), rule 15,
(8). $W \rightarrow (Z \rightarrow (W \wedge Z))$	axiom 6,
(9). $Z \rightarrow (W \wedge Z)$	by (4), (8), rule 15,
(10). $Y \rightarrow X$	via (1), (7),
(11). $Z \rightarrow X$	via (2), (9),
(12). $(Y \vee Z) \rightarrow X$	via (10, (11), axiom 3,
(13). X	by (5), (12), rule 15,

(14). $W \rightarrow X$ deduction theorem cancelling premise (4). [65]

***Lemma* 7.3:** If a does not occur in W, $Y(x)$ or X,

$$\frac{\vdash (W \wedge Y(a)) \rightarrow X,\ \vdash W \rightarrow (\exists x)Y(x)}{\vdash W \rightarrow X.}$$

Proof:

(1). $(W \wedge Y(a)) \rightarrow X$ by hypothesis, theorems,
(2). $W \rightarrow (\exists x)Y(x)$ by hypothesis, theorems,
(3). W premise,
(4). $(\exists x)Y(x)$ by (2), (3), rule 15,
(5). $W \rightarrow (Y(a) \rightarrow (W \wedge Y(a)))$ axiom 6,
(6). $Y(a) \rightarrow (W \wedge Y(a))$ by (3), (5), rule 15,
(7). $Y(a) \rightarrow X$ via (1), (6),
(8). $(\exists x)Y(x) \rightarrow X$ by (7), rule 13,
(9). X by (4), (8), rule 15,
(10). $W \rightarrow X$ deduction theorem cancelling premise (3).

§ 8. A second axiom system, $\mathcal{A}_2$

We introduce a second, very similar, axiom system, and prove equivalence. $\mathcal{A}_2$ has the same axioms as $\mathcal{A}_1$, as well as rules 13 and 14. It does not have rule 15. Instead it has rules

14a. $$\frac{X(a)}{(\forall x)X(x)}$$

15a. $$\frac{(\forall x_1) \ldots (\forall x_n)X,\ (\exists x_1) \ldots (\exists x_n)X \rightarrow Y}{Y}$$

provided all parameters of $(\forall x_1) \ldots (\forall x_n)X$ are also in Y (n may be 0).

To show the two systems are equivalent, it suffices to show 14a and 15a are derived rules of $\mathcal{A}_1$, and 15 is a derived rule of $\mathcal{A}$2.

To show 14a is a derived rule of $\mathcal{A}_1$, suppose in $\mathcal{A}_1$ we have $X(a)$.

Let T be any theorem of $\mathcal{A}_1$ with no parameters. By axiom 1, $X(a) \rightarrow (T \rightarrow X(a))$, so by rule 15, $T \rightarrow X(a)$. Since a is not in T, by rule 14, $T(\forall x)X(x)$. But also T, so by rule 15, $(\forall x)X(x)$.

To show 15a is a derived rule of $\mathcal{A}_1$, suppose in $\mathcal{A}_1$ we have $(\forall x_1) \ldots (\forall x_n)X(x_1, \ldots, x_n)$ and $(\exists x_1) \ldots (\exists x_n)X(x_1, \ldots, x_n) \rightarrow Y$, and all [66] parameters of $(\forall x_1) \ldots (\forall x_n)X(x_1, \ldots, x_n)$ are in Y. From $(\forall x_1) \ldots (\forall x_n)X(x_1, \ldots, x_n)$, by axiom 12, $X(a_1, \ldots, a_n)$. From axiom 11, $X(a_1, \ldots, a_n) \rightarrow (\exists x_1) \ldots (\exists x_n)X(x_1, \ldots, x_n)$, so by rule 15, $(\exists x_1) \ldots (\exists x_n)X(x_1, \ldots, x_n)$ and by rule 15 again, Y.

Finally to show rule 15 is a derived rule of $\mathcal{A}_2$, suppose we have X and $X \rightarrow Y$ in $\mathcal{A}_2$. Let $a_1, a_2, \ldots, a_n$ be those parameters of X not in Y. Since we have $X(a_1, \ldots, a_n)$, by rule 14a, $(\forall x_1) \ldots (\forall x_n)X(x_1, \ldots, x_n)$. Similarly, since $X(a_1, \ldots, a_n) \rightarrow Y$ and $a_1, \ldots, a_n$ do not occur in Y, by rule 13, $(\exists x_1) \ldots (\exists x_n)X(x_1, \ldots, x_n) \rightarrow Y$. Now by rule 16a, Y.

Thus $\mathcal{A}_1$ and d$\mathcal{A}_2$ are equivalent. For use in the next section we state the straightforward

***Lemma* 8.1:** If in $\mathcal{A}_2$ we can prove $X(a)$, there is a proof of the same length of $X(b)$ for any parameter b. (note: a does not occur in $X(b) = X(a)(^a{}_b)$).

9. Correctness of the system $\mathcal{A}_2$

***Theorem* 9.1:** If X is provable in $\mathcal{A}_2$, X is valid.

Proof: By induction on the length of the proof for X. If the proof is of length 1, X is an axiom and we leave the reader to show validity of the axioms.

Suppose the result is known for all formulas with proofs of length less than n steps, and X is provable in n steps. We investigate the steps involved in the proof of X. Axioms have been treated.

Suppose $X(a) \rightarrow Y$ in rule 13 is provable in less than n steps where a is not in Y. Then $X(a) \rightarrow Y$ is valid. Then $(\exists x)X(x) \rightarrow Y$ is provable. We wish to show it is valid. Take any model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and any $\Gamma \in \mathcal{G}$ and suppose $((\exists x)X(x) \rightarrow Y) \in \mathcal{P}(\Gamma)$. Suppose $\Gamma^* \models (\exists x)X(x)$. Then $\Gamma^* \models X(b)$ for some b. But $X(a) \rightarrow Y$ is provable, so by lemma 8.1 $(X(a) \rightarrow Y)(^a{}_b)$ is provable with a proof of the same length, hence by hypothesis, valid. Since a is not in Y, this is $X(b) \rightarrow Y$. By validity, $\Gamma^* \models X(b) \rightarrow Y$, hence $\Gamma^* \models Y$. Thus $\Gamma \models (\exists x)X(x) \rightarrow Y$.

Rules 14 and 14a are similar.

Rule 15a: Suppose $(\forall x_1)\ldots(\forall x_n)X$ and $(\exists x_1)\ldots(\exists x_n)X \rightarrow Y$ are both provable and valid. Then Y is provable. We wish to show Y is valid. Let $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ be any model and $\Gamma \in \mathcal{G}$. Suppose $Y \in \mathcal{P}(\Gamma)$. Then [67] $(\forall x_1)\ldots(\forall x_n)X$ and $(\exists x_1)\ldots(\exists x_n)X \rightarrow Y$ are both in $\mathcal{P}(\Gamma)$, and since they are valid, $\Gamma \models (\forall x_1)\ldots(\forall x_n)X$ and $\Gamma \models (\exists x_1)\ldots(\exists x_n)X \rightarrow Y$. By the latter, either $\Gamma =\!\mid (\exists x_1)\ldots(\exists x_n)X$ or $\Gamma \models Y$. If $\Gamma =\!\mid (\exists x_1)\ldots(\exists x_n)X$, for any $a_1, \ldots, a_n \in \mathcal{P}(\Gamma)$, $\Gamma =\!\mid X(a_1, \ldots, a_n)$ contradicting $\Gamma \models (\forall x_1)\ldots(\forall x_n)X$. Hence $\Gamma \models Y$.

§ 10. **Completeness of the system $\mathcal{A}_1$**

The following Henkin type proof was discovered independently by Thomason [21], Aczel [1], and the author.

We work in the system $\mathcal{A}_1$. Let Γ be a set of *unsigned* formulas and $\boldsymbol{P}$ a collection of parameters. Suppose all the parameters of Γ are among those in $\boldsymbol{P}$.

Definition 10.1: By the *deductive completion of* Γ with respect to $\boldsymbol{P}$ we mean the smallest set of formulas Δ involving only parameters of $\boldsymbol{P}$ such that for any X over $\boldsymbol{P}$

$$\Gamma \vdash X \Rightarrow X \in \Delta.$$

We call Γ *deductively complete with respect to* $\boldsymbol{P}$ if it is its own deductive completion with respect to P.

We say Γ has the *Or-property* if

$$X \vee Y \in \Gamma \Rightarrow X \in \Gamma \text{ or } Y \in \Gamma.$$

We say Γ has the $\exists$-property if, for some parameter a,

$$(\exists x)X(x) \in \Gamma \Rightarrow X(a) \in \Gamma.$$

We call F *nice with respect to P if*

(1). Γ is deductively complete with respect to P,

(2). Γ has the *Or*-property,

(3). Γ has the $\exists$-property,

(4). Γ is consistent.

Remark 10.2: Consistency here has its usual meaning.

***Lemma* 10.3:** Let Γ be a set of formulas and X a single formula. Let $\boldsymbol{P}$ be the set of all parameters of Γ or X. Let $\{a_1, a_2, a_3, \ldots\}$ be a countable [68] collection of distinct parameters not in $\boldsymbol{P}$, and let $\boldsymbol{Q} = \boldsymbol{P} \cup \{a_1, a_2, a_3, \ldots\}$. If $\Gamma \dashv X$, then Γ can be extended to a set Δ which is nice with respect to $\boldsymbol{Q}$ such that $X \notin \Delta$.

Proof: Let $Z_1, Z_2, Z_3, \ldots$ be an enumeration of all formulas with parameters from $\boldsymbol{Q}$ of the form $Y \vee Z$ or $(\exists x)Y(x)$.

Since $\Gamma \dashv X$, Γ is consistent. We define a sequence $\{\Gamma_n\}$ as follows:

Let Γ_0 be the deductive completion of Γ with respect to $\boldsymbol{P}$. Then Γ_0 is consistent and $\Gamma_0 \dashv X$. Suppose we have defined Γ_n so that Γ_n is deductively complete with respect to $\boldsymbol{P} \cup \{a_1, a_2, \ldots, a_n\}$ and $\Gamma_n \dashv X$. Let $\Delta_n^0 = \Gamma_n$.

Suppose we have defined Δ_n^j $(j < n)$ so that it is consistent and $\Delta_n^j \dashv X$.

Let $\Delta_n^{j+1} = \Delta_n^j$ if

(1) $Z_j \notin \Delta_n^j$, or

(2a) $Z_j \in \Delta_n^j$, $Z_j = Y \vee Z$ and $Y \in \Delta_n^j$ or $Z \in \Delta_n^j$, or

(2b) $Z_j \in \Delta_n^j$, $Z_j = (\exists x)Y(x)$ and $Y(a) \in \Delta_n^j$ for some a.

This leaves the two key cases:

(3). Suppose $Z_j \in \Delta_n^j$ and Z_j is $Y \vee Z$ but $Y \notin \Delta_n^j$, $Z \notin \Delta_n^j$. We claim we can add one of Y or Z to Δ_n^j so that the result still does not yield X. For otherwise

$$\begin{aligned} \Delta_n^j, Y &\vdash X \\ \Delta_n^j, Z &\vdash X \\ \Delta_n^j &\vdash Y \vee Z \end{aligned}$$

(since $Y \vee Z \in \Delta_n^j$). But then by lemma 7.2 $\Delta_n^j \vdash X$, a contradiction. So add to Δ_n^j one of Y or Z so that the result does not yield X. Call the result Δ_n^{j+1}.

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Commento:

(4). Suppose $Z_j \in \Delta_n^j$ and Z_j is $(\exists x)Y(x)$, but $Y(a) \notin \Delta_n^j$ for any a. Take the first unused a_i of $\{a_1, a_2, \ldots\}$. We claim we can add $Y(a_i)$ to Δ_n^j and the result will not yield X. This is as above but by lemma 7.3. Thus $\Delta_n^j, Y(a_i) \dashv X$. Let Δ_n^{j+1} be $\Delta_n^j, Y(a_i)$.

Thus in any case Δ_n^{j+1} is consistent and $X \notin \Delta_n^{j+1}$. Let Γ_{n+1} be the deductive completion of $\Delta^n{}_n$ with respect to $\boldsymbol{P} \cup \{a_1, a_2, \ldots, a_k\}$ where a_k is the last parameter used in $\Delta_n{}^n$. Let $\Delta = \cup\Gamma_n$, then Δ has the following properties:

Δ uses exactly the parameters of $\boldsymbol{Q}$.

$X \notin \Delta$ since $X \notin \Gamma_n$ for any n.

Δ is deductively complete with respect to $\boldsymbol{Q}$.

Δ has the *Or*-property. For if $Y \vee Z \in \Delta$, *say* $Y \vee Z = Z_n$, then $Y \vee Z \in \Delta_m$ for some m. We can take $m > n$. Then $Y \vee Z = Z_n \in \Delta_m{}^n$, so either Y or Z is in $\Delta_m{}^{n+1} \subseteq \Delta$.

Similarly, Δ has the $\exists$-property. [69]

***Lemma* 10.4:** If Γ is nice with respect to $\boldsymbol{P}$:

(1). $X \wedge Y \in \Gamma \Leftrightarrow X \in \Gamma$ and $Y \in \Gamma$,
(2). $X \vee Y \in \Gamma \Leftrightarrow X \in \Gamma$ or $Y \in \Gamma$,
(3). $\sim X \in \Gamma \Leftrightarrow X \notin \Gamma$,
(4). $X \rightarrow Y \in \Gamma \Leftrightarrow X \notin \Gamma$ or $Y \in \Gamma$,
(5). $(\exists x)X(x) \in \Gamma \Leftrightarrow X(a) \in \Gamma$ for some $a \in \boldsymbol{P}$,
(6). $(\forall x)X(x) \in \Gamma \Rightarrow X(a) \in \Gamma$ for every $a \in \boldsymbol{P}$.

Proof: (1). By axioms 4, 5 and 6, since Γ is deductively complete with respect to $\boldsymbol{P}$.

(2). $X \vee Y \in \Gamma \Rightarrow X \in \Gamma$ or $Y \in \Gamma$, since Γ has the *Or*-property. The converse holds by axioms 7 and 8.

(3). If $\sim X \in \Gamma$, $X \notin \Gamma$, since Γ is consistent (using axiom 9).

(4). If $X \rightarrow Y \in \Gamma$, either $X \notin \Gamma$ or $Y \in \Gamma$ since Γ is deductively complete with respect to $\boldsymbol{P}$.

(5). If $(\exists x)X(x) \in \Gamma$, $X(a) \in \Gamma$ for some $a \in \boldsymbol{P}$ since Γ has the $\exists$-property. The converse is by axiom 11.

(6). By axiom 12.

***Lemma* 10.5:** Suppose Γ is nice with respect to $\boldsymbol{P}$ and $\{a_1, a_2, a_3 ...\}$ is a set of distinct parameters not in $\boldsymbol{P}$. Let $\boldsymbol{Q} = \boldsymbol{P} \cup \{a_1, a_2, a_3 ... \}$. Then

(1). If X has all its parameters in $\boldsymbol{P}$ but $\sim X \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that $X \in \Delta$.

(2). If $X \rightarrow Y$ has all its parameters in $\boldsymbol{P}$ but $X \rightarrow Y \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that $X \in \Delta$ and $Y \notin \Delta$.

(3). If $X(x)$ has all its parameters in $\boldsymbol{P}$ but $(\forall x)X(x) \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that for some $a \in \boldsymbol{Q}$, $X(a) \notin \Delta$.

Proof:

(1). Since $\sim X \notin \Gamma$, $\{\Gamma, X\}$ is consistent, for otherwise $\Gamma, X \vdash \sim X$. So by the deduction theorem $\Gamma \vdash X \rightarrow \sim X$ and by axiom 10 $\Gamma \vdash \sim X$, so $\sim X \in \Gamma$. Since $\{FX\}$ *is* consistent, there is some Y such that $\Gamma, X \dashv Y$. Now use lemma 10.3.

(2). $\Gamma, X \dashv Y$ for otherwise, by the deduction theorem $\Gamma \vdash X \rightarrow Y$, so $X \rightarrow Y \in \Gamma$. Since $\Gamma, X \dashv Y$, use lemma 10.3.

(3). $a_1 \notin P$. We claim $\Gamma \dashv X(a_1)$. Suppose $\Gamma \vdash X(a_1)$. For the conjunction, call it W, of some finite subset of Γ, $\vdash W \rightarrow X(a_1)$. F W X(a1). But a_1 does not [70] occur in W. By rule 14 $\vdash W \rightarrow (\forall x)X(x)$, so $\Gamma \vdash (\forall x)X(x)$, $(\forall x)X(x) \in \Gamma$. Since $\Gamma \dashv X(a_1)$, use lemma 10.3.

Now we proceed to show completeness. We arrange the parameters as follows:

$$\begin{array}{l} \boldsymbol{S}_1\text{: } a_1^{\,1}, a_2^{\,1}, a_3^{\,1}, \ldots \\ \boldsymbol{S}_2\text{: } a_1^{\,2}, a_2^{\,2}, a_3^{\,2}, \ldots \\ \boldsymbol{S}_3\text{: } a_1^{\,3}, a_2^{\,3}, a_3^{\,3}, \ldots \\ \ldots \ldots \ldots \ldots \ldots \ldots \end{array}$$

and let $\boldsymbol{P}_n = S_1 \cup S_2 \cup \ldots \cup S_n$. Let $\mathcal{G}$ be the collection of all nice sets with respect to any $\boldsymbol{P}_i$. If $\Gamma \in \mathcal{G}$, Γ is nice with respect to, say, $\boldsymbol{P}_n$. Let $\mathcal{P}(\Gamma) = \boldsymbol{P}_n$. Let $\Gamma \mathcal{R} \Delta$ if $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$ and $\Gamma \subseteq \Delta$. For any X, let $\Gamma \models X$ iff $X \in \Gamma$. By lemmas 10.4 and 10.5 $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is a model.

Finally, suppose $\dashv X$. All the parameters are in, say, $\boldsymbol{P}_n$. Since $\varnothing \dashv X$, by lemma 10.3 we can extend $\varnothing$ to a set Γ, nice with respect to $\boldsymbol{P}_n$ such that $X \notin \Gamma$. Thus $\Gamma \in \mathcal{G}$, $X \in \mathcal{P}(\bar{\Gamma})$ and $\Gamma \,=\!\!| X$.

Remark 10.6: This is a "universal" model in the sense of § 6.

In ch. 6 § 4 we will show that the set of all theorems using only parameters of $\boldsymbol{P}_n$ is itself a nice set with respect to $\boldsymbol{P}_n$. This would make the final use of lemma 10.3 above unnecessary. [71]

CHAPTER 6

ADDITIONAL FIRST ORDER RESULTS

§ 1. Compactness

We call an infinite set $\boldsymbol{S}$ of signed formulas realizable if there is a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and a $\Gamma \in \mathcal{G}$ such that for any formula X

$$TX \in \boldsymbol{S} \Rightarrow X \in \mathcal{P}(\Gamma) \text{ and } \Gamma \models X,$$
$$FX \in \boldsymbol{S} \Rightarrow X \in \mathcal{P}(\Gamma) \text{ and } \Gamma =\!\mid X.$$

There is a similar concept for sets of unsigned formulas $\boldsymbol{U}$. We say U is satisfiable if there is a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and a $\Gamma \in \mathcal{G}$ such that for any formula X

$$X \in \boldsymbol{U} \Rightarrow X \in \mathcal{P}(\Gamma) \text{ and } \Gamma \models X.$$

***Lemma* 1.1:** Let $\boldsymbol{U}$ be a set of unsigned formulas and define a set $\boldsymbol{S}$ of signed formulas to be $\{TX \mid X \in \boldsymbol{U}\}$. Then
(1). $\boldsymbol{U}$ is satisfiable if and only if $\boldsymbol{S}$ is realizable,
(2). $\boldsymbol{U}$ is consistent if and only if $\boldsymbol{S}$ is consistent.

Proof: Part (1) is obvious. To show part (2), suppose U is not consistent. Then some finite subset $\{u_1, ..., u_n\}$ is not consistent, so from it we can deduce any formula. Let A be an atomic formula having no predicate symbols or parameters in common with $\{u_1, ..., u_n\}$. Then

$$\vdash_I (u_1 \wedge ... \wedge u_n) \rightarrow A.$$ [72]

Hence there is a closed tableau for

$$\{F(u_1 \wedge ... \wedge u_n) \rightarrow A\},$$

so there is a closed tableau for

$$\{T(u_1 \wedge ... \wedge u_n), FA\}.$$

By the way we have chosen A, there must be a closed tableau for $\{T(u_1 \wedge ... \wedge u_n)\}$ and hence for $\{Tu_1, ..., Tu_n\}$. Thus $\boldsymbol{S}$ is not consistent.
The converse is trivial.

Because we have this lemma, we will only discuss realizability and consistency of sets of signed formulas.

Lemma 1.2: Let $\boldsymbol{S}$ be a set of signed formulas. If $\boldsymbol{S}$ is realizable, $\boldsymbol{S}$ is consistent.

Proof: If S is not consistent, some finite subset $\boldsymbol{Q}$ is not consistent. That is, there is a closed tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which $\mathcal{C}_1$ is $\{\boldsymbol{Q}\}$. If $\boldsymbol{Q}$ were realizable, by theorem 5.2.7 every $\mathcal{C}_i$ would be, but a closed configuration is not realizable.

Lemma 1.3: Let $\boldsymbol{S}$ be a *finite* set of signed formulas. If $\boldsymbol{S}$ is consistent, $\boldsymbol{S}$ is realizable.

Proof: Let $\boldsymbol{S}$ be $\{TX_1, ..., TX_n, FY_1,..., FY_m\}$.

$\boldsymbol{S}$ is consistent if and only if

$$\{F(X_1 \wedge \ldots \wedge X_n) \rightarrow (Y_1 \vee \ldots \vee Y_m)\}$$

is consistent. If this is consistent, $(X_1 \wedge \ldots \wedge X_n) \rightarrow (Y_1 \vee \ldots \vee Y_m)$ is a non-theorem, so by the completeness theorem, there is a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and a $\Gamma \in \mathcal{G}$ such that $X_i \in \mathcal{P}^{-}(\Gamma)$, $Y_j \in \mathcal{P}^{-}(\Gamma)$ and

$$\Gamma =\!| \ (X_1 \wedge \ldots \wedge X_n) \rightarrow (Y_1 \vee \ldots \vee Y_m).$$

But then for some Γ

$$\Gamma^* \models (X_1 \wedge \ldots \wedge X_n),\ \Gamma^* =\!| \ (Y_1 \vee \ldots \vee Y_m),$$

so $\boldsymbol{\Gamma^*}$ realizes $\boldsymbol{S}$.

This method does not work if $\boldsymbol{S}$ is infinite, but the lemma remains true, at least for sets with no parameters. The result can be extended to sets with some parameters, but we will not do so. [73]

***Lemma* 1.4:** Let $\boldsymbol{S}$ be an infinite set of signed formulas with no parameters. If $\boldsymbol{S}$ is consistent, $\boldsymbol{S}$ is realizable.

Proof: The proof can be based on either of the two tableau completeness proofs.

If we use the first proof, that of ch. 5 § 5, change step 0 to: "$\boldsymbol{S}$ is consistent. Extend it to a Hintikka element with respect to $\mathcal{P}_1$. Call the result Γ_1". Continue the proof as written. The lemma is then obvious.

If we use the proof of ch. 5 § 6 the result is even easier. $\boldsymbol{S}$ is consistent, so by lemma 5.6.4, we can extend $\boldsymbol{S}$ to a set Γ which is good with respect to $\boldsymbol{P}_i$. The result follows immediately.

***Theorem* 1.5:** If $\boldsymbol{S}$ is any set of signed formulas with no parameters, $\boldsymbol{S}$ is consistent if and only if $\boldsymbol{S}$ is realizable.

***Corollary* 1.6:** If every finite subset of $\boldsymbol{S}$ is realizable, so is $\boldsymbol{S}$.

***Corollary* 1.7:** If $\boldsymbol{U}$ is any set of unsigned formulas with no parameters, $\boldsymbol{U}$ is consistent if and only if $\boldsymbol{U}$ is satisfiable.

Remark 1.8: The last corollary could have been established directly by adapting the completeness proof of ch. 5 § 10.

Definition 1.9: For a set of formulas $\boldsymbol{U}$, by $\Gamma \models \boldsymbol{U}$ we mean $\Gamma \models X$ for all $X \in \boldsymbol{U}$.

***Corollary* 1.10** (strong completeness): Let $\boldsymbol{U}$ be any set of unsigned formulas with no parameters. Then $\boldsymbol{U} \vdash_{\mathrm{I}} X$ if and only if in any model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$, for any $\Gamma \in \mathcal{G}$, if $\Gamma \models U$, $\Gamma \models X$.

Proof: $\boldsymbol{U} \vdash_{\mathrm{I}} X$ if and only if $\{TY \mid Y \in \boldsymbol{U}\} \cup \{FX\}$ is inconsistent.

Corollary 1.11: (cut elimination, Gentzen's Hauptsatz): If $\boldsymbol{S}$ is a set of signed formulas with no constants and $\{\boldsymbol{S}, TX\}$ and $\{S, FX\}$ are inconsistent, so is $\{S\}$.

Remark 1.12: This may be extended to sets $\boldsymbol{S}$ with some parameters. To be precise, to any set S which leaves unused a countable collection of parameters. It follows that in the completeness proof of ch. 5 § 6 a set 4 maximal consistent with respect to $\boldsymbol{P}$ actually contains TX or FX for each X with parameters from $\boldsymbol{P}$. [74]

§ 2. Concerning the excluded middle law

If $\boldsymbol{S}$ is a set of unsigned formulas, by $\boldsymbol{S} \vdash_{\mathrm{C}} X$ and $\boldsymbol{S} \vdash_{\mathrm{I}} X$ we mean classical and intuitionistic derivability respectively.

Let $X(\alpha_1, ..., \alpha_n)$ be a formula having exactly the parameters $\alpha_1, ..., \alpha_n$. By the closure of X we mean the formula

$$(\forall x_{i(1)}) \ldots (\forall x_{i(n)})\, X\, (x_{i(1)}, \ldots, x_{i(n)})$$

(where $x_{i(j)}$ does not occur in $X(\alpha_1, ..., \alpha_n)$.

Let $\boldsymbol{M}$ be the collection of the closures of all formulas of the form $X \vee \sim X$. We wish to show:

***Theorem* 2.1:** If X has no parameters,

$$\vdash_{\mathrm{C}} X \Leftrightarrow \boldsymbol{M} \vdash_{\mathrm{I}} X.$$

We first show:

***Lemma* 2.2:** Let $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ be a model, $\Gamma \in \mathcal{G}$, and suppose $Y \boldsymbol{M} \Rightarrow \Gamma \models Y$. Then Γ can be included in a complete $\mathcal{R}$-chain C such that C' is a truth set (see ch. 4 § 6).

Proof: Enumerate all formulas beginning with a universal quantifier: $X_1, X_2, X_3, \ldots$

Let $\Gamma_0 = \Gamma$. Having defined Γ_n, consider X_{n+1}. If $X_{n+1} \notin \mathcal{P}(\Gamma_n{*})$ for any $\Gamma_n{*}$, let $\Gamma_{n+1} = \Gamma_n$. Otherwise there is some $\Gamma_n{*}$ such that $X_{n+1} \in \mathcal{P}(\Gamma_n{*})$. Say X_{n+1} is $(\forall x)X(x)$. We have two cases:

(1). If $\Gamma_n{*} \models (\forall x)X(x)$, let $\Gamma_{n+1} = \Gamma_n{*}$.

(2). If $\Gamma_n{*} \mathrel{=\!|} (\forall x)X(x)$, there is a $\Gamma_n{**}$ and an $\alpha \in \mathcal{P}(\Gamma_n{**})$ such that $\Gamma_n{**} \mathrel{=\!|} X(\alpha)$. Let Γ_{n+1} be this $\Gamma_n{**}$.

Let the $\mathcal{R}$-chain $\mathcal{C}$ be $\{\Gamma_0, \Gamma_1, \Gamma_2, ...\}$. Since $Y \in \boldsymbol{M} \Rightarrow \Gamma \models Y$ and $\Gamma = \Gamma_0$, $\mathcal{C}$ is a complete $\mathcal{R}$-chain by the definition of $\boldsymbol{M}$, and so $\mathcal{C}'$ is an almost truth set. Thus we have only one more fact to show:

$$Y(\alpha) \in \mathcal{C}' \text{ for every parameter of } \mathcal{C}' \Rightarrow (\forall x)Y(x) \in \mathcal{C}'.$$

Suppose $(\forall x)Y(x, \alpha_1, \ldots, \alpha_n) \notin \mathcal{C}'$ (where $\alpha_1,..., \alpha_n$ are all the parameters of Y). If some α_i is not a parameter of $\mathcal{C}'$, we are done. So suppose each α_i occurs in $\mathcal{C}'$. Then for some $\Gamma_n \in \mathcal{C}$, all $\alpha_i \in \mathcal{P}(\Gamma_n)$ and $\Gamma_n =\!| (\forall x)Y(x, \alpha_1, \ldots, \alpha_n)$. But by the construction of $\mathcal{C}$, there is a Γ_m $(m \geq n)$ such that $\Gamma_m =\!| Y(b, \alpha_1, ..., \alpha_n)$ for some $b \in \mathcal{P}(\Gamma_m)$. But

$$\Gamma \models (\forall x_1) \ldots (\forall x_n) (\forall x) [Y(x, x_1, ..., x_n) \vee \sim \underline{Y}(x, x_1, ..., x_n)]$$ [75]

and $\Gamma \mathcal{R} \Gamma_m$, so

$$\Gamma_m \models Y(b, \alpha_1, \ldots, \alpha_n) \vee \sim Y(b, \alpha_1, \ldots, \alpha_n)$$

thus $\Gamma_m \models \sim Y(b, \alpha_1, \ldots, \alpha_n)$. $\sim Y(b, \alpha_1, \ldots, \alpha_n) \in \mathcal{C}'$, so $Y(b, \alpha_1, \ldots, \alpha_n) \notin \mathcal{C}'$ for a parameter b of $\mathcal{C}'$.

Now to prove the theorem itself:

If $\boldsymbol{M} \vdash_I X$ then for some finite subset $\{m_1, ..., m_n\}$ of $\boldsymbol{M}$

$$\vdash_I (m_1 \wedge \ldots \wedge m_n) \rightarrow X.$$

By theorem 4.8.2 (and the completeness theorems)

$$\vdash_C (m_1 \wedge \ldots \wedge m_n) \rightarrow X.$$

But $\vdash_C (m_1 \wedge \ldots \wedge m_n)$, hence $\vdash_C X$.

Conversely, if $\boldsymbol{M} \dashv_I X$, let $\boldsymbol{S}$ be the set of signed formulas

$$\{FX\} \cup \{TY \mid Y \in \boldsymbol{M}\}.$$

Since $\boldsymbol{M} \dashv_I X$, $\boldsymbol{S}$ is consistent. Then by the results of the last section, $\boldsymbol{S}$ is realizable. Thus there is a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ and a $\Gamma \in \mathcal{G}$ such that $Y \in \boldsymbol{M} \Rightarrow \Gamma \models Y$, $X \in \mathcal{P}(\Gamma)$ and $\Gamma \dashv X$. But X has no parameters, so $X \vee \sim X \in \boldsymbol{M}$. Thus $\Gamma \models X \vee \sim X$, so $\Gamma \models \sim X$. Now by lemma 2.2 there is a truth set containing $\sim X$. Hence $\dashv_C X$.

§ 3. Skolem-Löwenheim

By the domain of a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ we mean $\cup_{\Gamma \in \mathcal{G}} \mathcal{P}(\Gamma)$. So far we have only considered models in which the domain was at most countable. Suppose now we have an uncountable number of parameters and we change the definitions of formula, model and validity accordingly, but not the definition of proof.

***Theorem* 3.1:** X is valid in all models if and only if X is valid in all models with countable domains.

Proof. One half is trivial.

Suppose there is a model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ with an uncountable domain in which X is not valid. The correctness proof of ch. 5 § 2 or 9 is still applicable. Thus X is not provable. Since X is not provable, if we reduce the collection of parameters to a countable number (including those of X), X still will not be provable. Then any of the completeness proofs will furnish a counter-model for X with a countable domain. [76]

This method may be combined with that of § 1 to show

***Theorem* 3.2:** If $\boldsymbol{S}$ is any countable set of signed formulas with no parameters, $\boldsymbol{S}$ is consistent if and only if $\boldsymbol{S}$ is realizable in a model with a countable domain.

***Theorem* 3.3:** If $\boldsymbol{U}$ is any countable set of unsigned formulas with no parameters, $\boldsymbol{U}$ is consistent if and only if $\boldsymbol{U}$ is satisfiable in a model with a countable domain.

Remark 3.4: In part II, we will be using models with domains of arbitrarily high cardinality.

§ 4. Kleene tableaus

The system of this section is based on the intuitionistic system G3 of [10]. The modifications are due to Smullyan. The resulting system is like that of Beth except that sets of signed formulas never contain more than one F-signed formula. Explicitely, everything is as it was in ch. 2 § i and ch. 5 § 1 except that the reduction rules are replaced by the following, where $\boldsymbol{S}$ is a set of signed formulas with at most one F-signed formula.

$$KT\wedge:\ \frac{\boldsymbol{S},\ T(X \wedge Y)}{\boldsymbol{S},\ TX,\ TY} \qquad KF\wedge:\ \frac{\boldsymbol{S}_T,\ F(X \wedge Y)}{\boldsymbol{S}_T,\ FX} \quad \frac{\boldsymbol{S}_T,\ F(X \wedge Y)}{\boldsymbol{S}_T,\ FY}$$

$$KT\vee:\ \frac{\boldsymbol{S},\ T(X \vee Y)}{\boldsymbol{S},\ TX \mid \boldsymbol{S},\ TY} \qquad KF\vee:\ \frac{\boldsymbol{S}_T,\ F(X \vee Y)}{\boldsymbol{S}_T,\ FX,\ FY}$$

$$KT\sim:\ \frac{\boldsymbol{S},\ T(\sim X)}{\boldsymbol{S}_T,\ FX} \qquad KF\sim:\ \frac{\boldsymbol{S}_T,\ F(\sim X)}{\boldsymbol{S}_T,\ TX}$$

$$KT\rightarrow:\ \frac{\boldsymbol{S},\ T(X \rightarrow Y)}{\boldsymbol{S}_T,\ FX \mid \boldsymbol{S},\ TY} \qquad KF\rightarrow:\ \frac{\boldsymbol{S}_T,\ F(X \rightarrow Y)}{\boldsymbol{S}_T,\ TX,\ FY}$$

$$KT\exists:\ \frac{\boldsymbol{S},\ T(\exists x)X(x)}{\boldsymbol{S},\ TX(a)} \qquad KF\exists:\ \frac{\boldsymbol{S}_T,\ F\exists(x)X(x)}{\boldsymbol{S}_T,\ FX(a)}$$

$$KT\forall: \frac{\boldsymbol{S},\ T(\forall x)X(a)}{\boldsymbol{S},\ TX(a)} \qquad KF\forall: \frac{\boldsymbol{S_T},\ F(\forall x)X(a)}{\boldsymbol{S_T},\ FX(a)}$$

where in $KT\exists$ and $KF\forall$ the parameter a does not occur in S or $X(x)$. [77]

There are several ways of showing this is actually a proof system for intuitionistic logic. We choose to show it is directly equivalent to the Beth tableau system, that is, we give a proof translation procedure.

We leave it to the reader to show the almost obvious fact that anything provable by Kleene tableaus is provable by Beth tableaus. To show the converse, we need

***Lemma* 4.1:** If a Beth tableau for $\{TX_1, ..., TX_n, FY_1, ..., FY_m\}$ closes, then there is a closed Kleene tableau for

$$\{TX_1, ..., TX_n, F(Y_1 \vee ... \vee Y_m)\}.$$

Proof: The proof is by induction on the length of the closed Beth tableau. If the tableau is of length 1, the result is obvious. Now suppose we know the result for all closed Beth tableaus of length less than n, and a closed tableau for the set in question is of length n. We have several cases depending on the first step of the tableau.

If the first step is an application of rule $F\wedge$, the Beth tableau begins

$\{\{\boldsymbol{S_T}, FX_1, ..., FX_n, FY \wedge Z\}\}$,
$\{\{\boldsymbol{S_T}, FX_1, ..., FX_n, FY\}\},\{\{\boldsymbol{S_T}, FX_1, ..., FX_n, FZ\}\}$,

and proceeds to closure. Now by the induction hypothesis there are closed Kleene tableaus for $\{\{\boldsymbol{S_T},\ F(X_1 \vee \dots \vee X_n \vee Y)\}\}$, and $\{\{\boldsymbol{S_T},\ F(X_1 \vee \dots \vee X_n \vee Y)\}\}$. We have two possibilities:

(1). If Y is not "used" in the first tableau, or if Z is not "used" in the second tableau, a Kleene tableau beginning

$\{\{\boldsymbol{S_T},\ F(X_1 \vee \dots \vee X_n \vee (Y \wedge Z))\}\}$,
$\{\{\boldsymbol{S_T},\ F(X_1 \vee \dots \vee X_n)\}\}$,

must close.

(2). If both Y and Z are "used", a Kleene tableau beginning

$\{\{\boldsymbol{S_T},\ F(X_1 \vee \dots \vee X_n \vee (Y \wedge Z))\}\}$,
:
:
$\{\{\boldsymbol{S_T},\ F(Y \wedge Z)\}\}$,
$\{\{\boldsymbol{S_T},\ FY\}\}$, $\{\{\boldsymbol{S_T},\ FZ\}\}$,

must close.

The other cases are similar and are left to the reader. [78]

Thus the two tableau systems are equivalent. Now we verify a remark made at the end of eh. 5 § 10.

***Lemma* 4.2:** (Gödel, McKinsey and Tarski): $\vdash_I X \vee Y$ iff $\vdash_I X$ or $\vdash_I Y$.
Proof: Immediate from the Kleene tableau formulation.

***Lemma* 4.3:** (Rasiowa and Sikorski): If $\vdash_I (\exists x)X(x, a_1, ..., a_n)$ where $a_1, \ldots, a_n$ are all the parameters of X, then $\vdash_I X(b, a_1, ..., a_n)$ where b is one of the a_i. If X has no parameters, b is arbitrary and $\vdash_I (\forall x)X(x)$.
Proof: A Kleene tableau proof of $(\exists x)X(x, a_1, ..., a_n)$ begins

$$\{\{F(\exists x)X(x, a_1, ..., a_n)\}\},$$
$$\{\{(FX(b, a_1, ..., a_n)\}\}$$

and proceeds to closure. If b is some a_i, we are done. If not, we actually have a proof, except for a different first line, of

$$(\forall x)X(x, a_1, ..., a_n).$$

§ 5. Craig interpolation lemma

***Theorem* 5.1:** If $\vdash_I X \rightarrow Y$ and X and Y have a predicate symbol in common, then there is a formula Z involving only predicates and parameters common to X and Y such that $\vdash_I X \rightarrow Z$ and $\vdash_I Z \rightarrow Y$; if X and Y have no common predicates, either $\vdash_I \sim X$ or $\vdash_I Y$.

The classical version of this theorem was first proved by Craig, hence the name. The intuitionistic version is due to Schütte [17]. Essentially the same proof was given for a natural deduction system by Prawitz [15]. We give basically the same proof in the Kleene tableau system. For another proof in this system see [11].

We find it convenient to temporarily introduce two symbols t and f into our collection of logical symbols, letting them be atomic formulas, and letting them combine according to the following rules.

$X \vee t = t \vee X = t$,
$X \vee f = f \vee X = X$,
$X \wedge t = t \wedge X = X$,
$X \wedge f = f \wedge X = f$,
$\sim t = f$, $\sim f = t$, [79]
$X \rightarrow t = f \rightarrow X = t$,
$t \rightarrow X = X$ $X \rightarrow f = \sim X$,
$(\exists x)t = (\forall x)t = t$,
$(\exists x)f = (\forall x)f = f$.

By a *block* we mean a finite set of signed formulas containing at most one F-signed formula. When we call a block inconsistent, we mean there is a closed Kleene tableau for it. By an initial part of a block we mean any subset of the T-signed formulas. We make the convention that if $\boldsymbol{S}$ is the finite set of unsigned formulas $\{X_1, ..., X_n\}$ then $\boldsymbol{TS}$ is the set

$\{TX_1, ..., TX_n\}$. We further make the convention that for a set $\boldsymbol{S}$ of formulas, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ represent subsets such that $\boldsymbol{S}_1 \cap \boldsymbol{S}_2 = \varnothing$ and $\boldsymbol{S}_1 \cup \boldsymbol{S}_2 = \boldsymbol{S}$. By $[\boldsymbol{S}]$ we mean the set of predicates and parameters of formulas of $\boldsymbol{S}$, together with t and f.

Now we define an interpolation formula X for the block $\{T\boldsymbol{S}, FY\}$ (where $\boldsymbol{S}$ is a set of unsigned formulas and Y is a formula) with respect to the initial part $T\boldsymbol{S}_1$, which we denote by $\{T\boldsymbol{S}, FY\}/\{T\boldsymbol{S}_1\}$, as follows ($X$ may be t or f, but we assume t and f are not part of $\boldsymbol{S}$ or Y): X is an $\{T\boldsymbol{S}, FY\}/\{T\boldsymbol{S}_1\}$ if

(1). $[X] \subseteq [\boldsymbol{S}_1] \cap [\boldsymbol{S}_2, Y]$,

(2). $\{T\boldsymbol{S}_1, \mathrm{FX}\}$ is inconsistent,

(3). $\{TX, T\boldsymbol{S}_2, FY\}$ is inconsistent

(we have temporarily added to the closure rules: closure of a set of signed formulas if it contains Tf or Ft).

***Lemma* 5.2:** An inconsistent block has an interpolation formula with respect to every initial part.

Proof: We show this by induction on the length of the closed tableau for the block. If this is of length 1, the block must be of the form

$$\{T\boldsymbol{S}, TX, FX\}.$$

We have two cases:

Case (1). The initial part is $\{T\boldsymbol{S}_1, TX\}$. Then X is an interpolation formula.

Case (2). The initial part is $\{T\boldsymbol{S}_1\}$. Then $\{T\boldsymbol{S}_2, TX, FX\}$ is inconsistent and t is an interpolation formula.

Now suppose we have an inconsistent block, and the result is known for all inconsistent blocks with shorter closed tableaus. We have several cases depending on the first reduction rule used. [80]

KT ∨: The block is $\{T\boldsymbol{S}, TX \vee Y, FZ\}$, and $\{T\boldsymbol{S}, TX, FZ\}$ and $\{T\boldsymbol{S}, TY, FZ\}$ are both inconsistent.

Case (1). The initial part is $\{T\boldsymbol{S}_1, TX \vee \mathrm{Y}\}$. Then by the induction hypothesis there are formulas U_1 and U_2 such that

U_1 is an$\{T\boldsymbol{S}, TX, FZ\}/\{T\boldsymbol{S}1, TX\}$,

U_2 is an $\{T\boldsymbol{S}, TX, FZ\}/\{T\boldsymbol{S}1, TY\}$.

Then $U_1 \vee U_2$ is an $\{T\boldsymbol{S}, TX \vee Y, FZ)/\{T\boldsymbol{S}_1, TX \vee Y\}$.

Case (2). The initial part is $\{T\boldsymbol{S}_1\}$. Again, by hypothesis, there are U_1, U_2 such that

U_1 is an $\{T\boldsymbol{S}, TX, FZ\}/\{T\boldsymbol{S}_1\}$,

U_2 is an $\{T\boldsymbol{S}, TY, FZ\}/\{T\boldsymbol{S}_1\}$.

Then $U_1 \vee U_2$ is an $\{T\boldsymbol{S}, TX \vee Y, FZ\}/\{T\boldsymbol{S}1\}$.

KF∨: The block is $\{T\boldsymbol{S}, FX \vee Y\}$, and $\{T\boldsymbol{S}, FX\}$ or $\{TS, FY\}$ is inconsistent. Suppose the first. Let the initial part be $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, FX\}/\{T\boldsymbol{S}_1\}$.

Then U is an $\{TS, FX \vee Y\}/\{\boldsymbol{TS}_1\}$.

KT∧: The block is $\{\boldsymbol{TS}, TX \wedge Y, FZ\}$, and $\{\boldsymbol{TS}, TX, TY, FZ\}$ is inconsistent.

Case (1). The initial part is $\{\boldsymbol{TS}_1, TX \wedge Y\}$. By hypothesis there is a U such that

$$U \text{ is an } \{\boldsymbol{TS}, TX, TY, FZ\}/\{\boldsymbol{TS}_1, TX, TY\}.$$

Then U is an $\{\boldsymbol{TS}, TX \wedge Y, FZ\}/\{\boldsymbol{TS}_1, TX \wedge Y\}$.

Case (2). The initial part is $\{\boldsymbol{TS}_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{T\boldsymbol{S}, TX, TY, FZ\}/\{\boldsymbol{TS}_1\}.$$

Then U is an $\{\boldsymbol{TS}, TX \wedge Y, FZ\}/\{\boldsymbol{TS}_1\}$.

KF∧: The block is $\{\boldsymbol{TS}, FX \wedge Y\}$, and $\{\boldsymbol{TS}, FX\}$ and $\{\boldsymbol{TS}, FY\}$ are both inconsistent. Suppose the initial part is $\{\boldsymbol{TS}_1\}$. By hypothesis there are U_1, U_2 such that

$$U_1 \text{ is an } \{\boldsymbol{TS}, FX\}/\{\boldsymbol{TS}_1\},$$
$$U_2 \text{ is an } \{\boldsymbol{TS}, FY\}/\{\boldsymbol{TS}_1\}.$$

Then $U_1 \wedge U_2$ is an $\{\boldsymbol{TS}, FX \wedge Y\}/\{\boldsymbol{TS}_1\}$. [81]

KF~: The block is $\{\boldsymbol{TS}, F{\sim}X\}$, and $\{\boldsymbol{TS}, TX\}$ is inconsistent. Suppose the initial part is $\{\boldsymbol{TS}_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{\boldsymbol{TS}, TX\}/\{\boldsymbol{TS}_1\}.$$

Then U is an $\{\boldsymbol{TS}, F{\sim}X)/\{\boldsymbol{TS}_1\}$.

KT~: The block is $\{\boldsymbol{TS}, T{\sim}X, FY\}$, and $\{\boldsymbol{TS}, FX\}$ is inconsistent.

Case (1). The initial part is {TS,}. By hypothesis there is a U such that

$$U \text{ is an } (TS, FX\}/\{TS1)$$

Then U is an {TS, T"X, FY)/{TS1}.

Case (2). The initial part is $\{\boldsymbol{TS}_1, T{\sim}X\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX\}/\{TS2\}.$$

We claim that

$${\sim}U \text{ is an } \{\boldsymbol{TS}, T{\sim} X, FY\}/\{\boldsymbol{TS}_1\}.$$

First we verify its predicates and parameters are correct. By hypothesis $[U] \subseteq [\boldsymbol{S}_2] \cap [\boldsymbol{S}_1, X]$, so immediately $[{\sim}U] \subseteq [\mathbf{S}_1, {\sim}X] \cap [S_2, Y]$. We have the following two blocks are inconsistent:

$$\{\boldsymbol{TS}_2, FU\},$$
$$\{\boldsymbol{TS}_1, TU, FX\}.$$

It follows that the following two blocks are also inconsistent:

$\{T\boldsymbol{S}_1, T{\sim}X, F{\sim}U\}$,
$\{T\boldsymbol{S}_2, T{\sim}U, FY\}$,

and we are done.

KF→: The block is $\{TS, FX \rightarrow Y\}$, and $\{T\boldsymbol{S}, TX, FY\}$ is inconsistent. Suppose the initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, TX, FY\}/\{T\boldsymbol{S}_1\}$.

Then U is an $\{T\boldsymbol{S}, FX \rightarrow Y\}/\{T\boldsymbol{S}_1\}$.

KT→: The block is $\{TS, TX \rightarrow Y, FZ\}$, and $\{T\boldsymbol{S}, FX\}$ and $\{T\boldsymbol{S}, TY, FZ\}$ are both inconsistent.

Case (1). The initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there are U_1, U_2 such that

U_1 is an $\{T\boldsymbol{S}, FX\}/\{T\boldsymbol{S}_1\}$,
U_2 is an $\{T\boldsymbol{S}, TY, FZ)/\{T\boldsymbol{S}_1\}$, [82]

Then $U_1 \wedge U_2$ is an $\{T\boldsymbol{S}, TX \rightarrow Y, FZ\}/\{T\boldsymbol{S}_1\}$.

Case (2). The initial part is $\{T\boldsymbol{S}_1, TX \rightarrow Y\}$. By hypothesis there are U_1, U_2 such that

U_1 is an $\{T\boldsymbol{S}, FX\}/\{T\boldsymbol{S}_2\}$,
U_2 is an $\{T\boldsymbol{S}, TY, FZ\}/\{T\boldsymbol{S}_1, TY\}$.

We claim $U_1 \rightarrow U_2$ is an $\{T\boldsymbol{S}, TX \rightarrow Y, FZ\}/\{T\boldsymbol{S}_1, TX \rightarrow Y\}$.

By hypothesis

$[U_1] \subseteq [\boldsymbol{S}_2] \cap [\boldsymbol{S}_1, X]$,
$[U2] \subseteq [\boldsymbol{S}_1, Y] \cap [\boldsymbol{S}_2, Z]$,

so

$[U_1 \rightarrow U_2] \subseteq [\boldsymbol{S}_1, X \rightarrow Y] \cap [\boldsymbol{S}_2, Z]$.

We have that the following four blocks are inconsistent:

(1). $\{T\boldsymbol{S}_2, FU_1\}$,
(2). $\{TU_1, T\boldsymbol{S}_1, FX\}$,
(3). $\{T\boldsymbol{S}_1, TY, FU_2\}$,
(4). $\{TU_2, TS_2, FZ\}$,

and we must show the following two blocks are inconsistent:

$\{T\boldsymbol{S}_1, TX \rightarrow Y, FU_1 \rightarrow U_2\}$,
$\{TU_1 \rightarrow U_2, T\boldsymbol{S}_2, FZ\}$.

The first follows from (2) and (3), and the second from (1) and (4).

KF∃: The block is $\{T\boldsymbol{S}, F(\exists x)X(x)\}$, and $\{T\boldsymbol{S}, FX(a)\}$ is inconsistent. Suppose the initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, FX(a)\}/\{T\boldsymbol{S}_1\}$.

Then $[U] \subseteq [\boldsymbol{S}_1] \cap [\boldsymbol{S}_2, X(a)]$.

Case (1). $a \notin [U]$.

Then U is an $\{T\boldsymbol{S}, F(\exists x)X(x)\}/\{T\boldsymbol{S}_1\}$

Case (2). $a \in [U]$, $a \in [\boldsymbol{S}_2]$

Again U is an $\{T\boldsymbol{S}, F(\exists x)X(x)\}/(T\boldsymbol{S}_1\}$

Case (3). $a \in [U]$, $a \notin [\boldsymbol{S}_2]$.

Then $(\exists \mathrm{x})U(^a{}_x)$ is an

$$\{T\boldsymbol{S}, F(\exists x)X(x)\}/\{T\boldsymbol{S}_1\}.$$

KT∃: The block is $\{TS, T(\exists x)X(x), FZ\}$, and $\{TS, TX(a), FZ\}$ is inconsistent, where $a \notin [S, X(x), Z]$. [83]

Case (1). The initial part is $\{T\boldsymbol{S}_1, T(\exists x)X(x)\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, TX(a), FZ\}/\{T\boldsymbol{S}_1, TX(a)\}$.

Then U is an $\{T\boldsymbol{S}, T(\exists x)X(x), FZ\}/\{T\boldsymbol{S}_1, T(\exists x)X(x)\}$.

Case (2). The initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, TX(a), FZ\}/\{T\boldsymbol{S}_1\}$.

Then U is an $\{T\boldsymbol{S}, T(\exists x)X(x), FZ\}/\{T\boldsymbol{S}_1\}$.

$KF\forall$: The block is $\{T\boldsymbol{S}, F(\forall x)X(x)\}$, and $\{T\boldsymbol{S}, FX(a)\}$ is inconsistent, where $a \notin [\boldsymbol{S}, X(x)]$. Suppose the initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, FX(a)\}/\{T\boldsymbol{S}_1\}$.

Then U is an $\{\boldsymbol{S}, \mathrm{F}(\forall x)X(x)\}/\{T\boldsymbol{S}_1\}$.

$KT\forall$: The block is $\{T\boldsymbol{S}, \mathrm{T}(\forall x)X(x), FZ\}$, and $\{T\boldsymbol{S}, TX(a), FZ\}$ is inconsistent.

Case (1). The initial part is $\{T\boldsymbol{S}_1, T(\forall x)X(x)\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, TX(a), FZ\}/\{T\boldsymbol{S}_1, TX(a)\}$.

Case (la). $a \notin [U]$.
Then U is an

$\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1, T(\forall x)X(x)\}$.

Case (1b). $a \in [U]$, $a \in [\boldsymbol{S}_1, X(x)]$.
Again

U is an $\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1, T(\forall x)X(x)\}$.

Case (lc). $a \in [U]$, $a \notin [\boldsymbol{S}_1, X(x)]$.
Then $(\forall x)U(^a{}_x)$ is an $\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1, T(\forall x)X(x)\}$.

Case (2). The initial part is $\{T\boldsymbol{S}_1\}$. By hypothesis there is a U such that

U is an $\{T\boldsymbol{S}, TX(\mathrm{a}), FZ\}/\{T\boldsymbol{S}_1\}$.

Case (2a). $a \in [U]$.
Then U is an $\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1\}$.

Case (2b). $a \in [\mathrm{U}]$, $a \notin [\boldsymbol{S}_2, X(x), Z]$.
Again U is an $\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1\}$.

Case (2c). $a \in [U]$, $a \notin [\boldsymbol{S}_2, X(x), Z]$. [84]

Then $(\exists x)U(^a{}_x)$ is an $\{T\boldsymbol{S}, T(\forall x)X(x), FZ\}/\{T\boldsymbol{S}_1\}$.

Now to prove the original theorem 5.1:

Suppose $\vdash_{\mathrm{I}} X \rightarrow Y$. Then $\{TX, FY\}$ is inconsistent. By the lemma, there is a U such that U is an $\{TX, FY\}/\{TX\}$. We have three cases:

(1). $U = t$.
Then since $\{Tt, FY\}$ is inconsistent, $\vdash_{\mathrm{I}} Y$.

(2). $U = f$.
Then since $\{TX, Ff\}$ is inconsistent, $\{F{\sim}X\}$ is also inconsistent (f is not in X). Thus $\vdash_I {\sim}X$.

(3). $U \neq t$, $U \neq f$.
Then U is a formula not involving t or f, all the parameters and predicates of U are in X and Y, and since $\{TX, FU\}$ and $\{TU, FY\}$ are both inconsistent, $\vdash_I X \rightarrow U$ and $\vdash_I U \rightarrow Y$.

§ 6. Models with constant function

In part II we will be concerned with finding countermodels for formulas with no universal quantifiers, and we will confine ourselves to models with a constant function. To justify this restriction, we show in this section

***Theorem* 6.1**: If X is a formula with no universal quantifiers and $\dashv_I$ X, then there is a counter-model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ for X in which $\mathcal{P}$ is a constant function.

Definition 6.2: For this section only, let $a_1, a_2, a_3, \ldots$ be an enumeration of all parameters. We call a set Γ of signed formulas a Hintikka element if Γ is a Hintikka element with respect to some initial segment of $a_1, a_2, a_3, \ldots$ (see ch. 5 § 4).

***Lemma* 6.3**: If $\boldsymbol{S}$ is a finite, consistent set of signed formulas with no universal quantifiers, $\boldsymbol{S}$ can be extended to a *finite* Hintikka element.

Proof: Suppose $\boldsymbol{S}$ is the set $\{X_1, X_2, \ldots, X_n\}$ where each X_i is a *signed* formula. We define the two sequences $\{P_k\}$, $\{Q_k\}$ as follows:

Let

$$\boldsymbol{P}_0 = \varnothing,\ \boldsymbol{Q}_0 = X_1, \ldots, X_n.$$

Suppose we have defined $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ where

$$\boldsymbol{P}_k = Y_1, \ldots, Y_r,\ \ \boldsymbol{Q}_k = W_1, \ldots, W_s,$$ [85]

and $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$ (considered as a set) is consistent. To define $\boldsymbol{P}_{k+1}$ and $\boldsymbol{Q}_{k+1}$ we have several cases depending on W_1:

Case atomic: If W_1 is a signed atomic formula, let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, W_1,\ \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s.$$

Case T$\vee$: If W_1 is $TX \vee Y$, either TX or TY is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$, say TX. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, TX \vee Y,\ \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, TX.$$

Case F$\vee$: If W_1 is $FX \vee Y$ then FX, FY is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, FX \vee Y,\ \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, FX, FY.$$

Cases T$\wedge$, *F*$\wedge$, *T*$\sim$, *T*$\rightarrow$ are similar.

Case T$\exists$: If W_1 is $T(\exists x)X(x)$, let a be the first in the sequence $a_1, a_2, \ldots$ not occurring in $\boldsymbol{P}_k$ or $\boldsymbol{Q}_k$. Then $TX(a)$ is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, T(\exists x)X(x),\ \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, TX(a).$$

Case F∃: If W_1 is $F(\exists x)X(x)$, let $\{a_{i(1)},..., a_{i(t)}\}$ be the set of parameters occurring in $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$ such that no $FX(a_{i(j)})$ occurs in $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Then $\{FX(a_{i(1)}, ..., FX(a_{i(t)})\}$ is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_k, \boldsymbol{Q}_{k+1} = W_2,..., W_s, FX(a_{i(1)}),..., FX(ai(t), F(\exists x)X(x).$$

After finitely many steps there will be no T-signed formulas left in the $\boldsymbol{Q}$-sequence because each rule $T\vee$, $T\wedge$, $T\sim$, $T\rightarrow$, $T\exists$ reduces degree, and no rule $F\wedge$, $F\vee$, $F\exists$ introduces new T-signed formulas.

When no T-signed formulas are left in the Q-sequence, no new parameters can be introduced since rule $T\exists$ no longer applies. After finitely many more steps we must reach an unusable $\boldsymbol{Q}$-sequence. The corresponding $\boldsymbol{P} \cup \boldsymbol{Q}$-sequence is finite, consistent, and clearly a Hintikka element.

Remark 6.4: The above proof also shows the following which we will need later:
Let $\boldsymbol{R}$ be a finite Hintikka element. Suppose we add (consistently) a finite set of F-signed formulas to $\boldsymbol{R}$ and extend the result to a finite Hintikka element $\boldsymbol{S}$ by the above method. Then

$$\boldsymbol{R}_T = \boldsymbol{S}_T.$$ [86]

Since $\boldsymbol{R} \subseteq \boldsymbol{S}$, certainly $\boldsymbol{R}_T \subseteq \boldsymbol{S}_T$. That $\boldsymbol{S}_T \subseteq \boldsymbol{R}_T$. also holds follows by an inspection of the above proof; no new T-signed formulas will be added.

Now we turn to the proof of the theorem itself. We have no universal quantifiers to consider, so we may use the definition of associated sets in ch. 2 § 4.

Suppose X is a formula with no universal quantifiers, and $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to a finite Hintikka element $\boldsymbol{S}^0_0$. Let $T_1, \ldots, T_n$ be the associated sets of $\boldsymbol{S}$. Extend each to a finite Hintikka element, $\boldsymbol{S}^0_1,..., \boldsymbol{S}^0_n$ respectively. Thus we have

$$\boldsymbol{S}^0_0, \boldsymbol{S}^0_1,..., \boldsymbol{S}^0_n.$$

For each parameter a of some $\boldsymbol{S}^0_i$ and each formula of the form $F(\exists x)X(x)$ in $\boldsymbol{S}^0_0$, adjoin $FX(a)$ to $\boldsymbol{S}^0_0$ and extend the result to a Hintikka element $\boldsymbol{S}^1_0$. Do the same for $\boldsymbol{S}^0_1,..., \boldsymbol{S}^0_n$, producing $\boldsymbol{S}^1_1,..., \boldsymbol{S}^1_n$ respectively.
Thus we have now

$$\boldsymbol{S}^1_0, \boldsymbol{S}^1_1,..., \boldsymbol{S}^1_n.$$

Let $T_{n+1}, \ldots, T_m$ be the associated sets of $\boldsymbol{S}^1_0, \boldsymbol{S}^1_1,..., \boldsymbol{S}^1_n$. Extend each to a Hintikka element, $\boldsymbol{S}^0_{n+1}, ..., \boldsymbol{S}^0_m$ respectively. Thus we have now

$$\boldsymbol{S}^1_0, \boldsymbol{S}^1_1,..., \boldsymbol{S}^1_n, \boldsymbol{S}^0_{n+1}, ..., \boldsymbol{S}^0_m.$$

For each parameter a used so far, and for each formula of the form $F(\exists x)X(x)$ in $\boldsymbol{S}^1_0$ adjoin $FX(a)$ to $\boldsymbol{S}^1_0$ and extend the result to a finite Hintikka element $\boldsymbol{S}^2_0$. Do the same for each. Thus we have now

$$\boldsymbol{S}^2_0, \boldsymbol{S}^2_1,..., \boldsymbol{S}^2_n, \boldsymbol{S}^1_{n+1}, ..., \boldsymbol{S}^1_m.$$

Again take the associated sets, and extend to finite Hintikka elements, producing now

$$\boldsymbol{S}^2{}_0, \boldsymbol{S}^2{}_1, \ldots, \boldsymbol{S}^2{}_n, \boldsymbol{S}^1{}_{n+1}, \ldots, \boldsymbol{S}^1{}_m, \boldsymbol{S}^0{}_{m+1}, \ldots, \boldsymbol{S}^0{}_p.$$

Continue in this manner. Let

$$\boldsymbol{S}_0 = \bigcup_{k=0}^{\infty} \boldsymbol{S}^k{}_0$$

$$\boldsymbol{S}_1 = \bigcup_{k=0}^{\infty} \boldsymbol{S}^k{}_1$$

By the remark above, for each n,

$$\boldsymbol{S}_{nT} = \boldsymbol{S}^0{}_{nT} = \boldsymbol{S}^1{}_{nT} = \ldots$$

Thus if $\boldsymbol{S}^k{}_n$ has as an associated set S, $\boldsymbol{S}^j{}_m$, $\boldsymbol{S}_{nT} \subseteq \boldsymbol{S}_m$. [87]

It now follows that $\{\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots\}$ is a Hintikka collection. For example, suppose $F{\sim}Y \in \boldsymbol{S}_j$. Let k be the least integer such that $F{\sim}Y \in \boldsymbol{S}^k{}_j$.. By the above construction, there is some set $\boldsymbol{S}^0{}_r$ such that $\boldsymbol{S}^0{}_r$ is an associated set of $\boldsymbol{S}^k{}_j$ and $TY \in \boldsymbol{S}^0{}_r$. But then $\boldsymbol{S}^k{}_{jT} \subseteq \boldsymbol{S}^0{}_r$, so by the above $\boldsymbol{S}_{jT} \subseteq \boldsymbol{S}_r$, and $TY \in \boldsymbol{S}_r$. The other properties are shown similarly.

Moreover, $\mathcal{P}(\boldsymbol{S}_n) = \mathcal{P}(\boldsymbol{S}_m)$ for all m and n, as is easily seen. (Recall that $\mathcal{P}(\boldsymbol{S})$ is the collection of all parameters, used in $\boldsymbol{S}$.) Now as in ch. 5 § 3 there is a model for this Hintikka collection, and this model will have a constant map, so the theorem is shown. [88]

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